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Nonlinear Standing and Rotating Waves on the Sphere

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We establish the existence of time-periodic solutions of semi-linear wave equations on the unit sphere in $\mathbb{R}^3$. The problem has been studied previously in (1982, V. Benci and Fortunato, *Ann. Mat. Pura Appl.* **132**, 215–242) using variational techniques. Our results here are much sharper: We employ delicate methods of bifurcation theory (1979, H. Kielhöfer, *J. Math. Anal. Appl.* **68**, 408–420; 1987, H. Kielhöfer and P. Kötzner, *J. Appl. Math. Phys.* **38**, 201–212) combined with well-known group-theoretic ideas to find branches of small-amplitude solutions. Precise spatio-temporal patterns of solutions are uncovered as a by-product of the analysis. Moreover, in certain cases we prove the existence of global solution branches (in the sense of P. Rabinowitz). © 2000 Academic Press

1. INTRODUCTION

In this paper we establish the existence of time-periodic solutions of two-dimensional semi-linear wave equations of the form

$$\begin{aligned} Lu &\equiv u_{tt} - \Delta u = g(\mu, t, x, u) && \text{or} \\ Lu &\equiv u_{tt} + \Delta^2 u = g(\mu, t, x, u) && \text{on } \mathbb{R} \times S^2, \end{aligned} \tag{1.1}$$

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where $S^2 \subset \mathbb{R}^3$ denotes the unit sphere, "Δ" is the Laplace-Beltrami operator, "g" is a nonlinearity, and μ is a real parameter. We assume that $g(\mu, t, x, 0) \equiv 0$, and we look for free vibration solutions of (1.1), with $u(t+P, \cdot) \equiv u(t, \cdot)$, for some fixed period P, in a neighborhood of the equilibrium solution $u \equiv 0$.

Our goals here are similar to those of [6, 7] where the existence of such solutions was rigorously established for two-dimensional wave equations $(1.1)_2$ on square domains and on equilateral-triangular domains in $\mathbb{R}^2$. Namely, we employ delicate methods of local bifurcation theory [8] to find small-amplitude solutions. Detailed properties of solutions are subsequently obtained via well known generic, group-theoretic results [5, 14], viz., precise spatio-temporal patterns are uncovered on individual local solution branches. The rigorous use of such techniques in wave equations of the type (1.1) is not standard, and we refer the reader to the Introduction of [6] for a detailed discussion of the difficulties and their resolution. Moreover in this paper we can do much more: For $(1.1)_1$, and when $a \equiv g_u(0, t, x, 0)$ is an integer, we obtain global branches (in the sense of [15]) of periodic solutions. In specific examples, we further establish the spatio-temporal patterns of solutions on global continua.

Although the use of bifurcation theory for obtaining periodic solutions of nonlinear wave equations may seem natural (perhaps due its success in problems governed by ordinary differential equations—going back to [11]), in fact, it leads to difficult small-divisor problems, [8, 3]. Most known results from the literature are restricted to one-dimensional, semi-linear wave equations: In [2] the problem is treated "in the large", for fixed linear periods, via dual variational principles. Results for which the period is treated as a free parameter are obtained in [3] via Newton iteration in the spirit of KAM theory.

Our problem was also studied by Benci and Fortunato [1] (for $(1.1)_1$ on S^n—the unit sphere in $\mathbb{R}^{n+1}$) via dual variational methods, which were first employed in [2] for one-dimensional wave equations. In particular, they obtain critical points "in the large" for an associated indefinite functional. Accordingly their results are also global, i.e., the periodic solutions do not necessarily belong to a small neighborhood of $u \equiv 0$. However, no (spatio-temporal) structure of the solutions is revealed.

The outline of this paper is as follows: In Sections 2–4 we present an abstract summary of the methodology introduced in [8, 10], appropriate for a general class of problems, which includes (1.1). In Section 2 we study the abstract linear problem, introducing various function spaces and laying bare the basic ingredients needed to apply the methods of [8, 10]. In Section 3 we formulate the abstract nonlinear problem, and in Section 4 we provide bifurcation theorems for time-periodic solutions. These are of two types: bifurcation of paths of solutions and results for potential operators, the later type of which need not yield solution continua, cf. [9].

In Section 5 we give applications to (1.1). As in [6, 7, and 8], our use of bifurcation theory, and in particular, our ability to carry out a Liapunov–Schmidt reduction, depends crucially upon the following two properties: if $a \equiv g_u(0, t, x, 0)$ belongs to a certain subset of the rational numbers (dense in $\mathbb{R}$), then for corresponding "admissible linear" periods P, we have

$$0 < dim\ N(L - aI) < \infty, \tag{1.2}$$

and

$$(L - aI)^{-1} \text{ is bounded (on the orthogonal complement of the null space),} \tag{1.3}$$

where L is defined in (1.1). Properties (1.2), (1.3) enable a standard Liapunov reduction, i.e., no small-divisor problems arise. A general local bifurcation result for (1.1) is stated in Theorem 5.8.

In Section 6 we obtain general global bifurcation results for $(1.1)_1$ in the special case when $a \in \mathbb{Z}$, cf. Theorem 6.3. Then $(L - aI)^{-1}$ is not only bounded as in (1.3), but is also compact. We show by counter example that this compactness may fail when "a" is not an integer. Related compactness conditions are also at the heart of the results in [1]. To the best of our knowledge, this is the first application of global bifurcation theory to wave equations (see also Section 11).

Also, if we fix the period at $P = 2\pi$, then all bifurcation points $a \in \mathbb{Z}$ give rise to global solution continua of $(1.1)_1$, which may contain the solutions found in [1].

In Section 7 we establish notation for the systematic exploitation of symmetry. In Sections 8–10 we obtain detailed results on the spatio-temporal symmetries of solutions on local and global branches for cases when the nonlinearity possesses sufficient symmetries. Here we make use of the vast equivariant-normal-form calculations in [5, 14] for generic problems with S^1 temporal symmetries and $O(3)$ spatial symmetries. As in [6], our problem is also reversible in time, the additional exploitation of which enables isolation of solution branches via one-dimensional bifurcation analysis. In Section 8 we obtain standing waves, which are characterized by a fixed spatial symmetry while oscillating periodically in time. In Section 9 we find rotating waves, which correspond to rigidly rotating patterns. In both Sections 8 and 9, if $u \to g$ is odd, then we are also able to show that solutions on certain global continua have nodal curves in the form of great circles. (In Section 9, these great circles rotate.) Finally in Section 10 we get discrete-rotating waves—unlike rotating waves, these patterns do not rotate rigidly, but rather reappear in rotated form at regular fractions of the period.

2. THE LINEARIZED PROBLEM

In this section we investigate the linear problem

$$u_{tt} + Au - au = f \quad \text{in} \quad \mathbb{R} \times M,$$

$$u(t+P, x) = u(t, x) \quad \text{for all} \quad (t, x) \in \mathbb{R} \times M, \tag{2.1}$$

for some fixed period $P > 0$.

Here M is a two-dimensional, compact, connected, oriented Riemannian manifold without boundary. A is a real elliptic, self-adjoint, nonnegative operator of order $2m$, $m \geqslant 1$, and the number "a" is a real parameter. The function f is defined on the cylinder $Q_P \equiv \mathbb{R} \,(\mathrm{mod}\, P) \times M$. In this section we describe the appropriate functional analytic setting to solve (2.1). Since explicit examples are provided in Section 5 we confine ourselves here to the essential properties. Let $W^{2,k}(M) = H^k(M)$ with norm $\| \; \|_k$ denote the Hilbert spaces over M, where we employ the usual notation for Sobolev spaces, cf. [16]. We assume that

(2.2) A: $L^2(M) \to L^2(M)$ with domain of definition $D(A) = H^{2m}(M)$ is self-adjoint, nonnegative, and a Fredholm operator of index zero.

If M and the coefficients of A are smooth enough, we have $c_1 \|u\|_{2m+\ell} \leqslant (\|Au\|_\ell + \|u\|_0) \leqslant c_2 \|u\|_{2m+\ell}$ for $\ell \in \mathbb{N}_0$ and $u \in H^{2m+\ell}(M)$. Under these assumptions

(2.3) the operator A possesses a complete orthonormal system of eigenfunctions $\{\varphi_\ell\}$, $\ell \in \mathbb{N}_0$, in $L^2(M)$ with corresponding real eigenvalues λ_ℓ, each of finite multiplicity, $0 \leqslant \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, and $\lim_{\ell \to \infty} \lambda_\ell = \infty$.

Next we define the following real Hilbert spaces over the cylinder $Q_P \equiv \mathbb{R} \,(\mathrm{mod}\, P) \times M$:

$$H^{r,s}(Q_P) = H^r((0, P), H^0(M)) \cap H^0((0, P), H^s(M))$$

$$\text{for} \quad r, s \in \mathbb{N}_0, \tag{2.4}$$

(see [12], e.g.). Let for $(t, x) \in \mathbb{R} \times M$, $\ell \in \mathbb{N}_0$, $n \in \mathbb{Z}$,

$$\psi_{\ell n}(t,x)=\begin{cases}\dfrac{1}{\sqrt{P}}\varphi_\ell(x) & \text{for} \quad n=0,\\ \dfrac{2}{\sqrt{P}}\varphi_\ell(x)\cos\left(\dfrac{2\pi}{P}nt\right) & \text{for} \quad n<0,\\ \dfrac{2}{\sqrt{P}}\varphi_\ell(x)\sin\left(\dfrac{2\pi}{P}nt\right) & \text{for} \quad n>0,\end{cases} \tag{2.5}$$

be a complete orthonormal system in $L^2(Q_P)=H^{0,0}(Q_P)$. Then

$$H^{2,2m}(Q_P)=\Big\{u=\sum_{\substack{\ell\in\mathbb{N}_0\\ n\in\mathbb{Z}}} c_{\ell n}\psi_{\ell n} : c_{\ell n}\in\mathbb{R},$$

$$\|u\|^2_{2,2m}\equiv\sum c^2_{\ell n}(1+\lambda_\ell+n^2)^2<\infty\Big\}. \tag{2.6}$$

Definition (2.6), in turn, can be extended to the following:

$$H^{\alpha,\alpha m}(Q_P)=\Big\{u=\sum_{\substack{\ell\in\mathbb{N}_0\\ n\in\mathbb{Z}}} c_{\ell n}\psi_{\ell n} : c_{\ell n}\in\mathbb{R},$$

$$\|u\|^2_{\alpha,\alpha m}\equiv\sum c^2_{\ell n}(1+\lambda_\ell+n^2)^\alpha<\infty\Big\} \tag{2.7}$$

for real $\alpha\geqslant 0$.

It is easily seen that for $\alpha>\beta\geqslant 0$ the embedding $H^{\alpha,\alpha m}(Q_P)\subset H^{\beta,\beta m}(Q_P)$ is compact. Since $2m\geqslant 2$, $H^{2,2m}(Q_P)\subset H^2(Q_P)$ (continuously), and for $m=1$ the space $H^{\alpha,\alpha}(Q_P)=H^\alpha(Q_P)$ which is the usual Sobolev space of real order (see [1], Appendix 1, e.g.).

We now study the hyperbolic operator

$$Lu\equiv u_{tt}+Au. \tag{2.8}$$

If $u\in H^{2,2m}(Q_P)$, then

(2.9) $Lu=\sum_{l\in\mathbb{N}_0;\, n\in\mathbb{Z}} c_{\ell n}(\lambda_\ell-(4\pi^2/P^2)\,n^2)\,\psi_{\ell n}$ if u is given as in (2.6), and L: $H^{2,2m}(Q_P)\to L^2(Q_P)$ is continuous.

We define

(2.10) L: $H^{2,2m}(Q_P)\to H^{2,2m}(Q_P)$ with domain of definition $D(L)=\{u\in H^{2,2m}(Q_P) : Lu\in H^{2,2m}(Q_P)\}$.

Then

(2.11) L as defined in (2.10) is closed, and $D(L)$ with graph norm is a Hilbert space.

Now we are ready to formulate problem (2.1):

$$(L - aI)\, u = f, \qquad u \in D(L), \quad f \in H^{2,2m}(Q_P). \tag{2.12}$$

Let

$$S = \left\{ (\ell, n) \in \mathbb{N}_0 \times \mathbb{Z} : \lambda_\ell - a - \frac{4\pi^2}{P^2}\, n^2 = 0 \right\}. \tag{2.13}$$

Then the kernel of $L - aI$ is

$$N(L - aI) = \left\{ u = \sum_{(\ell, n) \in S} c_{\ell n} \psi_{\ell n} \right\} \subset D(L). \tag{2.14}$$

Later in Section 5 and 6, in the context of concrete examples, we shall see that the parameter "a" can be chosen so that the linear operator has two crucial properties, which we merely state here as hypotheses:

(2.15) $N(L - aI) \neq \{0\}$ is finite dimensional, i.e. $S = S_{a,P} = \{(\ell, n)$: $\lambda_\ell - a - (4\pi^2/P^2)\, n^2 = 0\}$ is nonempty and finite.

$|\lambda_\ell - a - (4\pi^2/P^2)\, n^2| \geqslant d$ for some $d > 0$ and for all $(\ell, n) \in (\mathbb{N}_0 \times \mathbb{Z}) \backslash S$.

Proposition 2.1. *Under the assumptions of this section and the hypotheses* (2.15), *the operator $L - aI$ with domain* (2.10) *is a Fredholm operator of index zero.*

For the proof we refer to [6, Prop. 2.1]. The same proof yields that

$$(L - aI)^{-1} \colon H^{2,2m}(Q_P) \cap N(L - aI)^\perp \to H^{2,2m}(Q_P) \cap N(L - aI)^\perp$$

is continuous. (2.16)

where $\perp$ refers to the scalar product in $L^2(Q_P)$.

3. THE NONLINEAR PROBLEM

We study the following nonlinear problem:

$$\begin{aligned} u_{tt} + Au - a(\mu) u &= h(\mu, t, x, u) && \text{on} \quad \mathbb{R} \times M, \\ u(t + P, x) &= u(t, x) && \text{for all} \quad (t, x) \in \mathbb{R} \times M. \end{aligned} \tag{3.1}$$

The function $a\colon \mathbb{R} \to \mathbb{R}$ is continuous, and for the nonlinearity $h\colon \mathbb{R} \times \mathbb{R} \times M \times \mathbb{R} \to \mathbb{R}$ we assume that

$$\begin{aligned} &h(\mu, t+P, x, u) = h(\mu, t, x, u), \\ &h, h_u \in C(\mathbb{R}, C^{2m}(\mathbb{R} \times M \times \mathbb{R})), \\ &h(\mu, t, x, 0) = h_u(\mu, t, x, 0) = 0 \qquad \text{for} \quad (\mu, t, x) \in \mathbb{R} \times \mathbb{R} \times M. \end{aligned} \tag{3.2}$$

Since M is two-dimensional, the lifting (2.10) of L from $L^2(Q_P)$ to $H^{2,2m}(Q_P)$ enables us to prove the following proposition:

PROPOSITION 3.1. *The substitution operator defined by*

$$H(\mu, u)(t, x) = h(\mu, t, x, u(t, x)) \tag{3.3}$$

maps $\mathbb{R} \times H^{2,2m}(Q_P)$ *continuously into* $H^{2,2m}(Q_P)$, *and it is continuously Frechet differentiable with respect to u such that*

$$H(\mu, 0) = 0, \qquad D_u H(\mu, 0) = 0. \tag{3.4}$$

The proof is similar to that in [6] and it is omitted here.

For our subsequent bifurcation analysis we rewrite problem (3.1) as

$$\begin{aligned} &G(\mu, u) \equiv (L - a(\mu)\, I)\, u - H(\mu, u) = 0, \\ &G\colon \mathbb{R} \times D(L) \to H^{2,2m}(Q_P), \end{aligned} \tag{3.5}$$

where $D(L) \subset H^{2,2m}(Q_P)$ (cf. (2.10)) is given the graph norm.

4. BIFURCATION

By virtue of (3.2) we have the trivial solution, i.e., $G(\mu, 0) = 0$ for all $\mu \in \mathbb{R}$. In this section we prove the existence of bifurcating solutions of (3.5) from the trivial line $\{(\mu, 0)\colon \mu \in \mathbb{R}\} \subset \mathbb{R} \times D(L)$. For convenience we define $a(0) = a$, and we henceforth assume that hypotheses (2.15) and (3.2) hold. The same arguments given in [6, Section 4], yield the following:

(4.1) G as given by (3.5) is continuous and continuously Frechet differentiable with respect to u, $D_u G(0, 0) = L - aI$ is a Fredholm operator of index zero, and 0 is an isolated eigenvalue of $D_u G(0, 0)$.

There is a continuous potential $g\colon \mathbb{R} \times D(L) \to \mathbb{R}$ which is differentiable with respect to u such that $D_u g(\mu, u) v = (G(\mu, u), v)_{L^2(Q_P)}$ holds for all $(\mu, u) \in \mathbb{R} \times D(L)$, $v \in D(L)$.

In order to apply the main theorem of [9] the computation of the so-called crossing number is in order. By definition (see [9]) the crossing number $\chi(D_u G(\mu, 0), 0)$ counts how many eigenvalues of $D_u G(\mu, 0) = L - a(\mu) I$ leave the left complex half-plane through 0 when the parameter μ varies from negative to positive values.

The eigenvalue of $D_u G(\mu, 0) = L - aI - (a(\mu) - a) I$ near zero is given by $a(\mu) - a$, having the multiplicity $\dim N(L - aI)$. Here the multiplicity is the geometric multiplicity, which is due to the fact that $L - aI$ is self-adjoint in $L^2(Q_P)$ (cf. (2.9)). Assuming that

$$a\colon \mathbb{R} \to \mathbb{R} \text{ is strictly monotonic for } \mu \text{ near } 0, \tag{4.2}$$

we have

$$\chi(D_u G(\mu, 0), 0) = \pm \dim N(L - aI) \neq 0. \tag{4.3}$$

Thus all conditions from [9] are fulfilled. Accordingly we have the following theorem.

THEOREM 4.1. *Under the hypotheses* (2.2), (2.15), (3.2), *and* (4.2) *the point* $(0, 0) \in \mathbb{R} \times D(L)$ *is a bifurcation point of nontrivial solutions of* $G(\mu, u) = 0$.

Theorem 4.1 is quite general in that the finite dimension of the kernel $N(L - aI)$ is arbitrary. However, it has two drawbacks: the bifurcating solutions need not form a curve or even a continuum in $\mathbb{R} \times D(L)$; the structure of the solutions (in terms of the modes (ℓ, n) in S, cf. (2.13)) is unknown. It simply guarantees that $(0, 0)$ is a cluster point of nontrivial solutions (μ, u) of (3.5).

However, if the kernel $N(L - aI)$ is two-dimensional, we get the following special case.

THEOREM 4.2. *If, under the assumptions of Theorem* 4.1, *we have* $\dim N(L - aI) = 2$, *i.e., if*

$$N(L - aI) = \mathit{span} \left\{ \varphi_{\ell_0}(x) \cos \frac{2\pi n_0}{P} t, \varphi_{\ell_0}(x) \sin \frac{2\pi n_0}{P} t \right\}, \tag{4.4}$$

if $a(\mu)$ *is continuously differentiable with* $a'(0) \neq 0$, *and if, in addition to* (3.2), *we have* $h_\mu, h_{\mu u} \in C(\mathbb{R}, C^{2m}(\mathbb{R} \times M \times \mathbb{R}))$, *then there is a continuously differentiable curve of nontrivial solutions of* (3.5), *denoted* $\{(\mu(\varepsilon), u(\varepsilon)) : |\varepsilon| < \delta\} \subset \mathbb{R} \times D(L)$, *emanating from the trivial solution branch at* $(\mu(0), u(0)) = (0, 0)$, *where*

$$u(\varepsilon)(t, x) = \varepsilon \varphi_{\ell_0}(x) \cos \frac{2\pi n_0}{P} t + o(|\varepsilon|) \tag{4.5}$$

and $o(|\varepsilon|)$ *is a perturbation in the* $H^{2,2m}(Q_P)$*-topology. Moreover,* $t \mapsto u(\varepsilon)(t, x)$ *is even, and all nontrivial solutions of* (3.5) *in a sufficiently small neighborhood of* (0, 0) *are given by this solution path* (*to within an arbitrary phase shift in* t).

Proof. Working in $\mathbb{R} \times \{u \in D(L) : u(-t, x) = u(t, x)\}$ the kernel of $D_u G(0, 0) = L - aI$ is one dimensional and the bifurcation theorem at simple eigenvalues [4] applies (see also [8, Theorem 2.4]). ▮

Remark 4.3. If $a = a(0)$ is an eigenvalue λ_ℓ of A, and if h does not depend on t, then we get bifurcation of stationary, i.e. time-independent solutions of (3.1) whether or not the hypotheses (2.15) are fulfilled. Indeed, by (2.2) the methods of [9] apply, since the crossing number is $\pm \dim N(A - aI) \neq 0$ in this case. However, in this paper we do not study these stationary solutions in detail.

Remark 4.4. Under the assumption of more regularity of the coefficients of A and of the function h (cf. (3.2)), the solutions of (3.5) in $\mathbb{R} \times D(L)$ are in fact classical solutions of (3.1). For a proof we refer to [6, Section 4].

5. EXAMPLES

In this section we explicitly verify our hypotheses of the first sections, in particular (2.2) and (2.15). We consider the Laplacian $A = -\Delta$ and the biharmonic operator $A = \Delta^2$ on $M = S^2$, which is the unit sphere in $\mathbb{R}^3$. Then it is well known that (2.2) is fulfilled with $m = 1$ and $m = 2$, respectively, and the eigenvalues are given by

$$\lambda_\ell = \ell(\ell + 1) \qquad \text{and} \qquad \lambda_\ell = (\ell(\ell + 1))^2, \tag{5.1}$$

respectively, for $\ell \in \mathbb{N}_0$.

The multiplicity of λ_ℓ is $2\ell + 1$ and the corresponding eigenfunctions are so-called spherical harmonics which are given in spherical coordinates $(x_1, x_2, x_3) = (\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha) \in S^2$ as follows:

$$\varphi_{\ell,h}(x_1, x_2, x_3) = \tilde{\varphi}_{\ell,h}(\alpha, \beta) = \begin{cases} \mathscr{P}_\ell(\cos\alpha) & \text{for} \quad h = 0, \\ \mathscr{P}_{\ell,h}(\cos\alpha)\cos h\beta & \text{for} \quad h > 0, \\ \mathscr{P}_{\ell,-h}(\cos\alpha)\sin(-h\beta) & \text{for} \quad h < 0, \end{cases} \tag{5.2}$$

$$\alpha \in [0, \pi], \qquad \beta \in [0, 2\pi), \qquad h = -\ell, \ldots, \ell, \qquad \ell \in \mathbb{N}_0.$$

The functions $\mathscr{P}_\ell$ are the Legendre polynomials and $\mathscr{P}_{\ell,h}$, $h=1, ..., \ell$, are the associated Legendre functions (see [5], e.g.). (There is an obvious change of notation in (5.1), (5.2) compared to (2.3))

First we address the hypotheses given in (2.15) for $A=-\Delta$.

PROPOSITION 5.1. *Let $a=p/q$, $p\in\mathbb{Z}$, $q\in\mathbb{N}$, $a\neq-\frac{1}{4}$, and let S (cf. (2.13)) be nonempty, i.e. $(\ell_0, n_0)\in S$ and assume that $n_0\neq 0$. If there is some $r_0\in\mathbb{N}$ such that*

$$q^2\lambda_{\ell_0}-pq=r_0^2, \tag{5.3}$$

then S is finite.

Proof. Since $(\ell_0, n_0)\in S$ with $n_0>0$ and $\lambda_{\ell_0}-a=r_0^2/q^2>0$ we have

$$P=P_0=\frac{2\pi n_0}{\sqrt{\lambda_{\ell_0}-a}}>0. \tag{5.4}$$

Then the characteristic equation defining the set S is equivalent to

$$q^2n_0^2\lambda_\ell-(q^2\lambda_{\ell_0}-pq)\,n^2=pqn_0^2, \tag{5.5}$$

and using $\lambda_\ell=\ell(\ell+1)$ and assumption (5.3) this becomes

$$\left(qn_0\left(\ell+\frac{1}{2}\right)-r_0n\right)\left(qn_0\left(\ell+\frac{1}{2}\right)+r_0n\right)=pqn_0^2+\frac{q^2n_0^2}{4}. \tag{5.6}$$

It suffices to restrict Eq. (5.6) to $n_0>0$ and to $(\ell, n)\in\mathbb{N}_0\times\mathbb{N}_0$. By assumption $a=p/q\neq-\frac{1}{4}$ the right hand side of (5.6) is nonzero and this implies that $|qn_0(\ell+\frac{1}{2})-r_0n|\geqslant\frac{1}{2}$ and

$$qn_0\left(\ell+\frac{1}{2}\right)+r_0n\leqslant 2\left(pqn_0^2+\frac{q^2n_0^2}{4}\right)=C_0 \tag{5.7}$$

for all $(\ell, n)\in S\cap(\mathbb{N}_0\times\mathbb{N}_0)$.

By $qn_0>0$ and $r_0>0$ (5.7) admits only finitely many solutions. We remark that (5.7) provides also an a priori estimate for $(\ell, n)\in S\cap(\mathbb{N}_0\times\mathbb{N}_0)$. ∎

Remark 5.2. If $a=-\frac{1}{4}$ then S is infinite if it is nonempty, i.e., if there is some $(\ell_0, n_0)\in\mathbb{N}_0\times\mathbb{Z}$ such that $P=2\pi n_0/(\ell_0+\frac{1}{2})$.

DEFINITION 5.3. For $A=-\Delta$ let the period P_0 be given as in (5.4). A rational number $a=p/q\neq-\frac{1}{4}$ is admissible for P_0 if (5.3) is fulfilled for some $r_0\in\mathbb{N}$. In this case $P_0=2\pi n_0q/r_0$, i.e., P_0 is a rational multiple of 2π.

PROPOSITION 5.4. *The set of all numbers* $a \in \mathbb{Q} \setminus \{-\frac{1}{4}\}$ *which are admissible for some period of the form* (5.4) *is dense in* $\mathbb{R}$.

The proof is similar to that in [6].

THEOREM 5.5. *For* $A = -\Delta$, *let* $a \in \mathbb{Q} \setminus \{-\frac{1}{4}\}$ *be admissible for some period* P_0 *of the form* (5.4). *Then both hypotheses given by* (2.15) *are fulfilled.*

Proof. For $P = P_0$ of the form (5.4) the pair (ℓ_0, n_0) is in S. By Proposition 5.1 admissibility of $a = \frac{p}{q} \neq -\frac{1}{4}$ for P_0 implies that S is finite. Finally, for $d = 1/n_0^2 q^2$

$$\left| \lambda_\ell - a - \frac{4\pi^2}{P_0^2} n^2 \right| = d \, |q^2 n_0^2 \lambda_\ell - r_0^2 n^2 - pqn_0^2| \geqslant d \tag{5.8}$$

for all $(\ell, n) \in (\mathbb{N}_0 \times \mathbb{Z}) \setminus S$. ∎

Next we take a different point of view: we prescribe $P = 2\pi/k$ for some fixed $k \in \mathbb{N}$ and we ask for which $a \in \mathbb{R}$ hypotheses (2.15) are fulfilled for $S = S_{a, 2\pi/k}$. Since the characteristic equation defining S is

$$\lambda_\ell - a - k^2 n^2 = 0 \qquad \text{for} \quad (\ell, n) \in \mathbb{N}_0 \times \mathbb{Z} \tag{5.9}$$

we see that $S \neq \emptyset$ only if $a \in \mathbb{Z}$.

THEOREM 5.6. *For a period* $P = 2\pi/k, k \in \mathbb{N}$, *and any* $a \in \mathbb{Z}$ *such that* $S_{a, P} \neq \emptyset$ *the hypotheses* (2.15) *are fulfilled.*

Proof. $S = S_{a, P}$ contains $(\ell, 0)$ if and only if $a = \lambda_\ell$, and if S contains any other (ℓ, n) such that $n > 0$ then (5.9) gives (5.3) with $a = p$, $q = 1$, and $r_0 = kn \in \mathbb{N}$. Therefore, by Proposition 5.1, S is finite. Furthermore, since $a \in \mathbb{Z}$, we have $|\lambda_\ell - a - k^2 n^2| \geqslant 1$ for all $(\ell, n) \in (\mathbb{N}_0 \times \mathbb{Z}) \setminus S$, i.e. we may choose $d = 1$ in (2.15). ∎

Remark 5.7. The number $a \in \mathbb{Z}$ is admissible for $P = 2\pi/k$ in the sense of Definition 5.3 if $S_{a, P}$ contains some (ℓ, n) with $n \neq 0$. If $S_{a, P} = \{(\ell, 0)\}$, which implies $a = \lambda_\ell$ is an eigenvalue of A, then a period P_0 of the form (5.4) does not exist for which "a" is admissible. Nonetheless the hypotheses (2.15) are fulfilled (cf. also Remark 4.3).

We are now ready to apply Theorem 4.1 to the problem

$$\begin{aligned} &u_{tt} - \Delta u = g(\mu, t, x, u) = a(\mu)\, u + h(\mu, t, x, u) \qquad \text{on} \quad \mathbb{R} \times S^2, \\ &u(t + P, x) = u(t, x). \end{aligned} \tag{5.10}$$

THEOREM 5.8. *Let* $h: \mathbb{R} \times \mathbb{R} \times S^2 \times \mathbb{R} \to \mathbb{R}$ *satisfy* (3.2) *with* $m=1$ *and assume that* $a: \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies* (4.2) *with* $a(0)=p/q \neq -\frac{1}{4}$ *admissible in the sense of Definition* 5.3 *for some period* P_0. *Then* $(\mu, u) = (0, 0)$ *is a bifurcation point of nontrivial solutions of* (5.10) *with* $P = P_0$ *in a weak sense, i.e.,* $(\mu, u) \in \mathbb{R} \times D(L)$. (*See also Remark* 4.4).

EXAMPLE 5.9. The number $a(0)=a=11$ is admissible for the period $P_0 = 2\pi$ with $r_0 = 1$. The solution set S of the characteristic equation is

$$\begin{aligned} &S = \{(3, 1), (4, 3), (11, 11), (3, -1), (4, -3), (11, -11)\} \qquad \text{and} \\ &\dim N(L - 11I) = 78, \end{aligned} \tag{5.11}$$

since the multiplicity of $\lambda_\ell = \ell(\ell+1)$ is $2\ell+1$.

For a proof use the estimate (5.7).

Next we investigate $A = \Delta^2$.

The eigenvalues λ_ℓ of the biharmonic operator $A = \Delta^2$ on $M = S^2$ are squares of an integer, cf. (5.1). Therefore the characteristic equation defining the set S is treated in the same way as in [6]. We summarize: If a rational number $a = p/q \neq 0$ is admissible for P_0 in the sense of Definition 5.3, then the set S is finite. With that modification Theorem 5.8 holds for

$$\begin{aligned} &u_{tt} + \Delta^2 u = g(\mu, t, x, u) = a(\mu)\, u + h(\mu, t, x, u) \qquad \text{on} \quad \mathbb{R} \times S^2, \\ &u(t+P, x) = u(t, x), \end{aligned} \tag{5.12}$$

if h satisfies (3.2) with $m=2$.

In order to apply Theorem 4.2 to (5.10) and (5.12), the dimension of the null space must be reduced. A first step in that direction uses the idea of a minimal period.

If not stated otherwise, we assume that $A = -\Delta$ or $A = \Delta^2$ in the sequel.

PROPOSITION 5.10. *Let* $a = p/q$ *be admissible for some period* P_0, *with "a" not an eigenvalue of* A. *Then the set of all periods for which "a" is also admissible contains a minimal period,* $P_1 = 2\pi/\sqrt{\lambda_{\ell_1} - a}$, *where* λ_{ℓ_1} *is an eigenvalue of* A. *Moreover, for* $P = P_1$, $\dim N(L - aI)$ *is equal to twice the multiplicity of* λ_{ℓ_1} *as an eigenvalue of* A, *i.e.,* $N(L - aI) = span\{\varphi_{\ell_1}(x) \times \cos(2\pi/P)\,t, \varphi_{\ell_1}(x) \sin(2\pi/P)\,t : (A - \lambda_{\ell_1} I)\varphi_{\ell_1} = 0\}$. *In view of* (5.2), $\dim N(L - aI) = 2(2\ell_1 + 1)$.

The proof is the same as that of Proposition 5.5 in [6].

The next proposition is converse to Proposition 5.10.

PROPOSITION 5.11. *Let $\lambda_{\ell_1} > 0$ be an eigenvalue of A. Then there exists an admissible number "a" for the period $P_1 = 2\pi/\sqrt{\lambda_{\ell_1} - a}$, which is minimal among all periods for which the number "a" is also admissible.*

Proof. First let $A = -\Delta$ and $\lambda_{\ell_1} = \ell_1(\ell_1 + 1)$, $\ell_1 \in \mathbb{N}$. We choose $a = p = \ell_1$ (and $q = 1$). Then "a" is admissible for P_1 with $r_1 = \ell_1 \in \mathbb{N}$. Assume that there is some smaller period P_2 for which "a" is admissible. Then the eigenvalue λ_{ℓ_2} defining P_2 is greater than λ_{ℓ_1} and $\ell_2(\ell_2 + 1) - \ell_1$ is a square, contradicting $\ell_2^2 < \ell_2(\ell_2 + 1) - \ell_1 < (\ell_2 + 1)^2$.

If $A = \Delta^2$ then $\lambda_{\ell_1} = (\ell_1(\ell_1 + 1))^2$ and $a = 2\sqrt{\lambda_{\ell_1}} - 1$ is admissible for P_1 with $r_1 = \ell_1(\ell_1 + 1) - 1 \in \mathbb{N}$. The assumption that there is some smaller period for which "a" is admissible, too, is contradictory, as shown in the proof of Proposition 5.6 in [6]. ∎

Remark 5.12. The choice $a = \ell_1$ for $A = -\Delta$ does not necessarily fulfill the additional requirement of Proposition 5.10. Obviously $\ell_1 = \ell(\ell + 1)$ is an eigenvalue of A for any $\ell \in \mathbb{N}$. But there is a second choice of "a" in Proposition 5.11 for $A = -\Delta$ such that "a" is not an eigenvalue of A: the number $a = (8\ell_1 - 1)/4$ is admissible for P_1 with $r_1 = 2(2\ell_1 - 1)$. We omit the calculation that P_1 is minimal.

The choice $a = 2\sqrt{\lambda_{\ell_1}} - 1$ is always good for $A = \Delta^2$: the eigenvalues $(\ell(\ell + 1))^2$ of A are even whereas $a = 2\sqrt{\lambda_{\ell_1}} - 1$ is odd and therefore not an eigenvalue of A.

Remark 5.13. Let $a = \frac{p}{q}$ be admissible for some period P_0 with "a" not an eigenvalue of A. If $(\ell_1, \pm n_1) \in S_{a, P_0}$ and if n_1 does not divide all other n for which $(\ell, n) \in S_{a, P_0}$ then for $P = P_1 = P_0/n_1 = 2\pi/\sqrt{\lambda_{\ell_1} - a}$, the same conclusion as in Proposition 5.10 holds. A simple argument proves that $S_{a, P_1} = \{(\ell_1, \pm 1)\}$. For the maximal $|n_1|$ we obtain the minimal period of Proposition 5.10.

If we choose the parameters $a = a(0)$ and $P = P_1$ according Propositions 5.10, 5.11, or Remark 5.13 we cannot yet apply Theorem 4.2 because the multiplicity of λ_{ℓ_1} is $2\ell_1 + 1 > 1$ for $\ell_1 \in \mathbb{N}$. Of course this degeneracy is due to spherical symmetry, the exploitation of which we postpone until Section 7, where Theorem 4.2 is applied.

6. GLOBAL BIFURCATION FOR THE NONLINEAR WAVE EQUATION

In this section we obtain *global* bifurcation results [15] for the nonlinear wave equation (3.1) on the sphere S^2 with $A = -\Delta$. For this we require an additional hypothesis:

(6.1) $a = a(0)$ is an integer and admissible for some period $P_0 = 2\pi/\sqrt{\lambda_{\ell_0} - a}$, and $S_{a,P_0} = \{(\ell_0, \pm n_0)\}$ for some $(\ell_0, n_0) \in \mathbb{N}_0 \times \mathbb{N}$.

If we choose the minimal period according Proposition 5.10, or if we obtain a period as in Remark 5.13, then (6.1) can be fulfilled for any admissible integer "a" which is not an eigenvalue of A.

PROPOSITION 6.1. *For any number $b \in \mathbb{R}\backslash\mathbb{Z}$, the operator $L + bI : D(L) \subset H^{2,2}(Q_P) \to H^{2,2}(Q_P)$, with $P = P_0$ as given in assumption* (6.1), *is bijective, and $(L+bI)^{-1} : H^{2,2}(Q_P) \to H^{2,2}(Q_P)$ is compact.*

Proof. By virtue of (2.6) and (2.9) we have

$$(L + bI)\,u = \sum_{\substack{\ell \in \mathbb{N}_0 \\ n \in \mathbb{Z}}} c_{\ell n} \left(\lambda_\ell + b - \frac{4\pi^2}{P^2}\, n^2 \right) \psi_{\ell n}, \tag{6.2}$$

and (6.1) implies that $P = 2\pi/r_0$ for some $r_0 \in \mathbb{N}$. Since $\lambda_\ell \in \mathbb{N}_0$ and $(4\pi^2/P^2)\, n^2 = r_0^2 n^2 \in \mathbb{N}_0$, the assumption $b \in \mathbb{R}\backslash\mathbb{Z}$ implies the injectivity of $L + bI$. The Fredholm property of $L + bI$ (cf. the proof of Proposition 2.1) then yields bijectivity and the continuity of $(L + bI)^{-1}$, cf. (2.16). We now sharpen that result as follows: Let $f \in H^{2,2}(Q_P)$, $f = \sum_{\ell \in \mathbb{N}_0;\, n \in \mathbb{Z}} d_{\ell n} \psi_{\ell n}$, and $(L + bI)\,u = f$. Then

$$u = \sum_{\substack{\ell \in \mathbb{N}_0 \\ n \in \mathbb{Z}}} c_{\ell n} \psi_{\ell n} \qquad \text{with} \quad c_{\ell n} = d_{\ell n}/(\lambda_\ell + b - r_0^2 n^2). \tag{6.3}$$

The proof of Lemma 4.3 in [1] implies that

$$|\lambda_\ell - r_0^2 n^2| \geqslant \tfrac{1}{2}\,(\ell + r_0\,|n|) \qquad \text{for all} \quad \ell \in \mathbb{N}_0, \quad n \in \mathbb{Z}. \tag{6.4}$$

Since $|b| \leqslant \frac{1}{4}(\ell + r_0\,|n| - 1)$ for all but finitely many $\ell \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ we get $|\lambda_\ell + b - r_0^2 n^2| \geqslant \frac{1}{2}(\ell + r_0\,|n|) - \frac{1}{4}(\ell + r_0\,|n| - 1) = \frac{1}{4}(1 + \ell + r_0\,|n|)$ for almost all $\ell \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ and (see (2.6), (2.7))

$$\begin{aligned} \|u\|_{3,3}^2 &= \sum_{\substack{\ell \in \mathbb{N}_0 \\ n \in \mathbb{Z}}} d_{\ell n}^2 \frac{(1 + \lambda_\ell + n^2)^3}{(\lambda_\ell + b - r_0^2 n^2)^2} \\ &\leqslant C_1 \sum d_{\ell n}^2 (1 + \lambda_\ell + n^2)^2 \frac{1 + \lambda_\ell + n^2}{(1 + \ell + r_0\,|n|)^2} \\ &\leqslant C_2\, \|f\|_{2,2}^2. \end{aligned} \tag{6.5}$$

Since the embedding $H^{3,3}(Q_P) \subset H^{2,2}(Q_P)$ is compact (cf. Section 2) the proof of Proposition 6.1 is complete. ∎

Remark 6.2. We now show by a counterexample that the compactness of $(L+bI)^{-1}$ is lost if we give up assumption (6.1). Choose $a=a(0)=p/q=\frac{7}{4}$. Then, by $q^2\lambda_1-pq=16\cdot 2-7\cdot 4=2^2=r_0^2$, the number "$a$" is admissible for $P_0=2\pi/\sqrt{\lambda_1-\frac{7}{4}}=4\pi$ which is minimal. We choose $f=\sum_{\ell\in\mathbb{N}_0;\, n\in\mathbb{Z}} d_{\ell n}\psi_{\ell n}$ such that $d_{\ell n}=0$ for all $n\neq 2\ell+1$ and $d_{\ell,\,2\ell+1}\neq 0$ for infinitely many $\ell\in\mathbb{N}_0$. Then, for $n=2\ell+1$, $\lambda_\ell+b-(4\pi^2/P_0^2)\,n^2=\ell^2+\ell+b-\frac{1}{4}(2\ell+1)^2=b-\frac{1}{4}$. Accordingly, from (6.3) we find $u=f/(b-\frac{1}{4})$ for any $b\neq\frac{1}{4}$; in particular, u has precisely the same regularity as f.

Next we rewrite (3.5) as follows:

$$\begin{aligned} G(\mu,u)\equiv(L-a(\mu)\,I)\,u-H(\mu,u)=0 &\Leftrightarrow\\ (L+bI)\,u-((a(\mu)+b)\,u+H(\mu,u))=0 &\Leftrightarrow\\ u-(L+bI)^{-1}\,((a(\mu)+b)\,u+H(\mu,u))=0 &\Leftrightarrow\\ u-F(\mu,u)=0 \qquad \text{for} \quad (\mu,u)\in\mathbb{R}\times H^{2,2}(Q_P). & \end{aligned} \tag{6.6}$$

In view of Propositions 3.1 and 6.1, we conclude that the mapping $F\colon\mathbb{R}\times H^{2,2}(Q_P)\to H^{2,2}(Q_P)$ as defined in (6.6) has the following property:

(6.7) $F\colon\mathbb{R}\times H^{2,2}(Q_P)\to H^{2,2}(Q_P)$ is completely continuous.

Thus the hypotheses of Rabinowitz' global bifurcation alternative are fulfilled (cf. [15]) provided the Leray–Schauder index of $I-F(\mu,\cdot)$ or of $I-D_uF(\mu,0)$ changes at $\mu=0$. Under assumption (6.1) we have a $(2\ell_0+1)$-dimensional kernel of $I-D_uF(0,0)=I-(L+bI)^{-1}\,(a(0)+b)I$ when we restrict (6.6) for $P=P_0$ to $\mathbb{R}\times\{u\in H^{2,2}(Q_P): u(-t,x)=u(t,x)\}$. If the function "$a$" is strictly monotonic for μ near 0 (cf. (4.2)), due to the odd-dimensional kernel the Leray–Schauder index changes at $\mu=0$, and we can state the following theorem.

THEOREM 6.3. *Let the integer $a=a(0)$ be admissible for some period $P=P_0$ such that (6.1) is fulfilled. Assume furthermore the hypotheses of Theorem 4.1. Then the point $(0,0)\in\mathbb{R}\times H^{2,2}(Q_P)$ is a bifurcation point of a continuum C_P of solutions of (6.6), and therefore of (5.10), having at least one of the following properties:*

(i) *C_P is unbounded in $\mathbb{R}\times H^{2,2}(Q_P)$ or*

(ii) *C_P is connected to some $(\mu_0,0)\neq(0,0)$ in $\mathbb{R}\times H^{2,2}(Q_P)$.*

All solutions on C_P are even functions of $t\in\mathbb{R}$ (mod P), and arbitrary phase shifts in t create new solutions connected to C_P.

COROLLARY 6.4. *Under the assumptions of Theorem* 6.3, *if* $0 < |a(0) - a(\mu)| < 1$ *for* $\mu \in \mathbb{R}\backslash\{0\}$ *then the global continuum fulfills alternative* (i): C_P *is unbounded in* $\mathbb{R} \times H^{2,2}(Q_P) \to H^{2,2}(Q_P)$.

Proof. If C_P meets the trivial solution line at some $(\mu_0, 0)$ then (by the implicit function theorem) necessarily $\dim N(L - a(\mu_0) I) > 0$. But $b = -a(\mu_0) \in \mathbb{R}\backslash\mathbb{Z}$ implies by Proposition 6.1 that $L - a(\mu_0)I$ is bijective such that alternative (ii) is ruled out. ∎

Remark 6.5. The same proof as for Proposition 6.1 shows also that

$$(L + bI)^{-1}: H^{\alpha,\alpha}(Q_P) \to H^{\alpha,\alpha}(Q_P) \tag{6.8}$$

is compact for all $\alpha \geqslant 0$ (cf. (2.7)).

If the function h defining the nonlinearity H (cf. (3.2)) possesses additional regularity, then Eq. (6.6), $u - F(\mu, u) = 0$, is well defined for $(\mu, u) \in \mathbb{R} \times H^{\alpha,\alpha}(Q_P)$ for $\alpha > 2$. By Sobolev's embedding theorem, the solutions given by Theorem 6.3 in $\mathbb{R} \times H^{\alpha,\alpha}(Q_P)$ are classical solutions of (5.10) or (5.12) provided $\alpha > 7/2$ (recall that $\dim M = 2$). Thus $h, h_u, h_\mu \in C(\mathbb{R}, C^4(\mathbb{R} \times S^2 \times \mathbb{R}))$ are sufficient to obtain classical solutions of (5.10).

EXAMPLE 6.6. We now apply our general results to the problem

$$\begin{aligned} u_{tt} - \Delta u &= \mu u + h(\mu, x, u) \qquad \text{on} \quad \mathbb{R} \times S^2, \\ u(t + P, x) &= u(t, x). \end{aligned} \tag{6.9}$$

For $a(\mu) \equiv \mu$ we obtain clearly $a(0) = 0$, which is too restrictive. Accordingly we change our notation slightly (via a parameter shift) and call $\mu = a$ a bifurcation point of (6.9) if, in our previous framework, $(\mu, u) = (0, 0)$ is a bifurcation point for (5.10) with $a(\mu) \equiv \mu + a$.

If we admit all "linear" periods $P = P_0$ of the form (5.4) then, by Proposition 5.4, Theorem 5.5, and Theorem 4.1, the entire real line consists of bifurcation points of (6.9), in this sense that each $(a, 0) \in \mathbb{R} \times D(L)$ is a cluster point of nontrivial solutions of (6.9) for some linear period.

Next we fix the period at $P = 2\pi$. By Theorem 5.6 the hypotheses (2.15) are fulfilled provided $S_{a, 2\pi} \neq \emptyset$, which, in turn, is true for $a = \lambda_\ell - n^2$ for any $(\ell, n) \in \mathbb{N}_0 \times \mathbb{Z}$, i.e., for all $a \in \mathbb{Z}$. Thus, by an application of Theorem 4.1,

(6.10) $\mathbb{Z}$ is the set of all bifurcation points of (6.9) for $P = 2\pi$.

We get more information about the nature of the bifurcating solutions when we apply Proposition 5.10. According to Remark 5.7, if $a = \lambda_\ell - n^2$ is not an eigenvalue of $A = -\Delta$, then Proposition 5.10 guarantees a minimal period $P_1 = 2\pi/\sqrt{\lambda_{\ell_1} - a} = 2\pi/r_1$ for some $r_1 \in \mathbb{N}$. Recall that $a \in \mathbb{Z}$, and without loss of generality, $q = 1$ in (5.3). Admissibility then implies $\lambda_{\ell_1} - a = r_1^2$.

On the other hand, by Remark 5.7 the number $a \in \mathbb{Z}$ is admissible for any period $2\pi/k$ provided $S_{a, 2\pi/k} \neq \emptyset$. Therefore $P_1 = 2\pi/r_1$ is also the minimal period among all periods $2\pi/k$ such that $S_{a, 2\pi/k} \neq \emptyset$. Obviously P_1-periodicity implies 2π-periodicity. Theorem 6.3 then yields:

(6.11) if $a \in \mathbb{Z}$ is not an eigenvalue of $A = -\Delta$ then "a" is a bifurcation point of a global solution continuum of (6.9) having period $P_1 = 2\pi/r_1$ which is the minimal period in $N(L - aI)$ with $D(L) \subset H^{2,2}(Q_{2\pi})$.

Finally, if $a \in \mathbb{Z}$ is an eigenvalue $\lambda_\ell = \ell(\ell + 1)$ of $A = -\Delta$, then we can improve Remark 4.3: due to the odd dimension, $2\ell + 1$, of $N(A - aI)$ we can apply the global result of [15] for stationary solutions. To summarize,

(6.12) all bifurcation points $a \in \mathbb{Z}$ give rise to global solution continua of (6.9) for $P = 2\pi$.

Finally, using the proof of Proposition 5.11, if we choose $a = \ell_1 \in \mathbb{N}$, and if ℓ_1 is not an eigenvalue of $A = -\Delta$, then global solution continua of (6.9) with period $P_1 = 2\pi/\ell_1$ bifurcate at $(\ell_1, 0)$. Moreover, the period $P_1 = 2\pi/\ell_1$ is minimal in $N(L - \ell_1 I)$. We illustrate this for Example 5.9: The minimal period in $N(L - aI)$ is $2\pi/11$, and at $a = 11$ a global continuum of $2\pi/11$-periodic solutions bifurcates.

The period $2\pi/3$ is not minimal in $N(L - 11I)$ for Example 5.9, but by an application of Remark 5.13, a global continuum of $2\pi/3$-periodic solutions bifurcates at $a = 11$, too.

Finally we mention the results of [1] for problem (6.9) with $P = 2\pi$. The existence results there are obtained by variational methods such that a parameter μ is not needed. On the other hand the conditions on the nonlinearity h are much more restrictive. A direct comparison of the results in [1] and our results is therefore not possible. Quite likely the solutions found in [1] are on our global solution continua.

7. EXPLOITATION OF SYMMETRY

We continue our study of (3.1) on $M = S^2$ with $A = -\Delta$ or $A = \Delta^2$. These problems admit certain symmetries the exploitation of which help to decrease the dimension of the kernel $N(L - aI)$ and hence enable a detailed local analysis. More on our use of group theory and the related notation may be found in [5] or briefly in [13], Appendix A.

Let $O(3)$ be the orthogonal group on $\mathbb{R}^3$ which leaves the unit sphere S^2 invariant. For all functions defined on S^2 we have the natural action of any $\sigma \in O(3)$ defined by

$$\sigma u(t, x) \equiv u(t, \sigma^{-1} x), \qquad (t, x) \in \mathbb{R} \ (\text{mod } P) \times S^2. \tag{7.1}$$

The actions

$$\begin{aligned} T_\kappa u(t, x) &\equiv u(t-\kappa, x) \qquad \text{for all} \quad \kappa \in \mathbb{R} \ (\text{mod } P), \\ Eu(t, x) &\equiv u(-t, x) \end{aligned} \tag{7.2}$$

together with (7.1) then define a representation of $O(2) \times O(3)$ on $H^{2,2m}(Q_P)$. Rewriting (5.10) or (5.12) abstractly via (3.5) as $G(\mu, u) = 0$ we can easily etablish the equivariance of $G(\mu, \cdot)$ with respect to any subgroup Γ of $O(2) \times O(3)$ as follows:

PROPOSITION 7.1. *Let Γ be a subgroup of $O(2) \times O(3)$ and $\Sigma_\Gamma \subset O(3)$ be its projection on $O(3)$. If h as given by* (3.2) *does not depend upon t and if h is invariant under Σ_Γ, i.e.*

$$h(\mu, \sigma x, u) = h(\mu, x, u) \qquad \textit{for all} \quad \sigma \in \Sigma_\Gamma, \quad x \in S^2, \quad \mu, u \in \mathbb{R}, \tag{7.3}$$

then $G(\mu, \cdot)$: $D(L) \subset H^{2,2m}(Q_P) \to H^{2,2m}(Q_P)$ is Γ-equivariant, i.e.

$$G(\mu, \gamma u) = \gamma G(\mu, u) \qquad \textit{for all} \quad \gamma \in \Gamma, \quad (\mu, u) \in \mathbb{R} \times D(L). \tag{7.4}$$

DEFINITION 7.2. Let Γ be a subgroup of $O(2) \times O(3)$. The fixed-point space of Γ is defined by

$$Fix_\Gamma(H^{2,2m}(Q_P)) \equiv \{u \in H^{2,2m}(Q_P) : \gamma u = u \text{ for all } \gamma \in \Gamma\}. \tag{7.5}$$

Since $O(2) \times O(3)$ is compact, it is well known that $Fix_\Gamma(H^{2,2m}(Q_P))$ is itself a Hilbert space with the inner product inherited from $H^{2,2m}(Q_P)$. Moreover, a direct consequence of (7.4) is that

$$G: \mathbb{R} \times (D(L) \cap Fix_\Gamma(H^{2,2m}(Q_P))) \to Fix_\Gamma(H^{2,2m}(Q_P)). \tag{7.6}$$

In other words, we can systematically find solutions of $G(\mu, u) = (L - a(\mu)\, I)\, u - H(\mu, u) = 0$ having specified symmetries by restricting G as in (7.6). Clearly, the Frechet derivative $D_u G(0, 0) = L - aI$ maps $D(L) \cap Fix_\Gamma(H^{2,2m}(Q_P))$ into $Fix_\Gamma(H^{2,2m}(Q_P))$, and by equivariance (7.4) of $G(\mu, \cdot)$ it commutes with any $\gamma \in \Gamma$. This, in turn, implies that the Fredholm property of $D_u G(0, 0)$ is preserved, and (4.1) is valid for the mapping G as given by (7.6). Applying our general bifurcation results to the restriction (7.6) we obtain the following result:

THEOREM 7.3. *Assume the hypotheses of Theorems* 4.1, *or* 4.2 *with*

$$\dim(N(L-aI)\cap Fix_{\Gamma}(H^{2,2m}(Q_P)))>0 \qquad or$$

$$\dim(N(L-aI)\cap Fix_{\Gamma}(H^{2,2m}(Q_P)))=1, \qquad respectively.$$

Then nontrivial solutions or a curve of solutions, respectively, of $G(\mu, u)=0$ *bifurcate from the trivial solution at* (0, 0). *If the hypotheses of Theorem* 6.3 *hold, and if*

$$\dim(N(L-aI)\cap Fix_{\Gamma}(H^{2,2}(Q_P))) \text{ is odd,}$$

then the bifurcating solutions form a global continuum.

In all cases the bifurcating solutions are contained in $\mathbb{R}\times Fix_{\Gamma}(H^{2,2m}(Q_P))$.

Remark 7.4. The restriction to the even functions in the proofs of Theorems 4.2 and 6.3 can now be restated from a symmetric point of view as follows: For $\Gamma'=\{I, E\}\times\{I\}\cong\mathbb{Z}_2\times\{I\}\subset O(2)\times O(3)$ we get

$$Fix_{\Gamma'}(H^{2,2m}(Q_P))=\{u\in H^{2,2m}(Q_P): u(-t, x)=u(t, x)\}, \tag{7.7}$$

i.e. we assume $\Gamma'\subset\Gamma$ in the last two cases of Theorem 7.3. The reduction of the dimension of the null space of $L-aI$ to one, however, requires additional spatial symmetries in the group Γ.

All solutions (μ, u) given by Theorem 7.3 are in $\mathbb{R}\times Fix_{\Gamma}(H^{2,2m}(Q_P))$. Therefore, the isotropy subgroup of these solutions, defined by

$$\Sigma_u=\{\gamma\in O(2)\times O(3): \gamma u=u\}, \tag{7.8}$$

clearly contains the group Γ but equality $\Gamma=\Sigma_u$ does not necessarily hold.

For local solutions (in a sufficiently small neighborhood of the bifurcation point), something more definitive can be said, which we now summarize below.

All our local bifurcation results (Theorems 4.1 and 4.2), including a local version of Theorem 6.3, are obtained by a Lyapunov–Schmidt reduction and then by solving the so-called bifurcation equation in the finite-dimensional subspace $N(L-aI)$.

Applying this method to the equivariant-problem (3.5), (7.4), it is well known that one can choose an equivariant Lyapunov–Schmidt reduction yielding a mapping F: $\mathbb{R}\times N(L-aI)\to N(L-aI)$ (defined only locally near (0, 0)) which is equivariant in the sense of (7.4) as well. Defining the fixed-point space of a subgroup $\Gamma\subset O(2)\times O(3)$ in $N(L-aI)$ as in (7.5), the equivariance of $F(\mu, \cdot)$ implies that

$$F: \mathbb{R}\times Fix_{\Gamma}(N(L-aI))\to Fix_{\Gamma}(N(L-aI)). \tag{7.9}$$

(Recall that F is defined only locally near $(0, 0)$). A complete picture of all local solutions of $G(\mu, u) = 0$ near $(0, 0)$ is then obtained from the finite-dimensional equivariant problem $F(\mu, v) = 0$. Since the isotropy of $v \in N(L - aI)$ is the same as the isotropy of $u \in H^{2,2m}(Q_P)$ (due to the equivariant Lyapunov–Schmidt reduction) we can apply all (local) results for the finite-dimensional problem and transfer it to the original infinite-dimensional equation. In particular, any local solution of $G(\mu, u) = 0$ in $\mathbb{R} \times (D(L) \cap Fix_\Gamma(H^{2,2m}(Q_P)))$ is obtained by solving $F(\mu, v) = 0$ in $\mathbb{R} \times Fix_\Gamma(N(L - aI))$. The existence of solutions is summarized in Theorem 7.3 for the cases that dim $Fix_\Gamma(N(L - aI))$ is one, odd, or arbitrary.

Remark 7.5. Locally near the bifurcation point $(0, 0)$, the bifurcating solutions of $G(\mu, u) = 0$ in $\mathbb{R} \times (D(L) \cap Fix_\Gamma(H^{2,2m}(Q_P)))$ obtained by Theorem 7.3 have the isotropy $\Sigma_u = \Gamma$ if Γ is a maximal isotropy subgroup of $O(2) \times O(3)$ in the natural representation (7.1), (7.2) on $N(L - aI)$, cf. [5].

The most important class of such maximal isotropy subgroups acting on $N(L - aI)$ is given by these subgroups Γ such that dim $Fix_\Gamma(N(L - aI)) = 1$. We take up this strategy in the next sections and we give examples of Γ yielding one-dimensional fixed-point subspaces of $N(L - aI)$. For that purpose we need a few definitions in order to characterize the structure of $N(L - aI)$.

We define for fixed $P > 0$, $n \in \mathbb{N}_0$, and $\ell \in \mathbb{N}_0$

$$U_{\ell, n} = span\left\{\varphi_{\ell, h}(x) \cos\frac{2\pi}{P} nt, \varphi_{\ell, h}(x) \sin\frac{2\pi}{P} nt : h = -\ell, ..., \ell\right\}, \tag{7.10}$$

where the functions $\varphi_{\ell, h}$ are given in (5.2). For later use we also define

$$V_\ell = span\{\varphi_{\ell, h}(x) : h = -\ell, ..., \ell\} = N(A - \lambda_\ell I), \tag{7.11}$$

cf. (5.1), (5.2). Observe that $V_\ell = U_{\ell, 0}$.

For the sake of a shorter notation we introduce the abbreviation $\mathcal{N}_0 = N(L - aI)$.

We conclude from (2.15) that $\mathcal{N}_0$ is the direct sum

$$\mathcal{N}_0 = \bigoplus_{(\ell, \pm n) \in S_{a, P}} U_{\ell, n}. \tag{7.12}$$

Furthermore, for $\Gamma \subset O(2) \times O(3)$ we have

$$Fix_\Gamma(\mathcal{N}_0) = \bigoplus_{(\ell, \pm n) \in S_{a, P}} Fix_\Gamma(U_{\ell, n}). \tag{7.13}$$

This is a consequence of the fact, that the action of $O(2)\times O(3)$ on $U_{\ell,n}$ given by (7.1), (7.2) leaves $U_{\ell,n}$ invariant. Clearly, if $\dim Fix_{\Gamma}(\mathcal{N}_0)=1$, then $Fix_{\Gamma}(\mathcal{N}_0)\subset U_{\ell_1,n_1}$ for some $(\ell_1,n_1)\in S_{a,P}$.

8. STANDING WAVES

In this and the following sections we study (3.1) on $M=S^2$ with $A=-\Delta$. The analogous results for (5.12), i.e., $A=\Delta^2$, are then readily obtained. The eigenfunctions are the same; only the parameters "a" and the period "P" require suitable adjustment. However, the global results, presented here and in Sections 9 and 10, hold for $A=-\Delta$ only, (cf. Section 6).

Here we investigate subgroups $\Gamma\subset O(2)\times O(3)$ of the form $\Gamma=D_n\times\Sigma$ where $\Sigma\subset O(3)$ is a subgroup. If

$$\dim(Fix_{\Sigma}(V_{\ell_1}))=d \qquad \text{for some} \quad \ell_1\in\mathbb{N}_0, \tag{8.1}$$

then we obtain

$$\dim(Fix_{D_{n_1}\times\Sigma}(\mathcal{N}_0))=d \qquad \text{for some} \quad n_1\in\mathbb{N}, \tag{8.2}$$

provided "a" is not an eigenvalue of A (recall that $\mathcal{N}_0=N(L-aI)$ is given by (7.12)). To see this, first suppose that $P=P_1$ is the minimal period, as given by Proposition 5.10. Then $\mathcal{N}_0=U_{\ell_1,1}$, i.e., $n_1=1$, and $D_1:=\tilde{\mathbb{Z}}_2$ is the two-element group $\tilde{\mathbb{Z}}_2=\{I,E\}$, where E is the time-reversal defined in $(7.2)_2$. As in (7.7), the elements of $Fix_{\tilde{\mathbb{Z}}_2\times\Sigma}(\mathcal{N}_0)$ are, in particular, even functions of "t". On the other hand, if we choose the period P_1 according to Remark 5.13, then P_1 need not be minimal. In this case, it is more convenient to work directly with the admissible period $P=P_0$ as follows: For all natural numbers n_1 such that $(\ell_1,n_1)\in S_{a,P}$ which do not divide any other n for which $(\ell,n)\in S_{a,P}$, we obtain

(8.3) $Fix_{\hat{\Gamma}}(\mathcal{N}_0)=U_{\ell_1,n_1}$ for $\hat{\Gamma}=\mathbb{Z}_{n_1}\times\{I\}$ with the action $T_{n(P/n_1)}u(t,x)=u(t-n(P/n_1),x)$ for $T_{n(P/n_1)}\in\mathbb{Z}_{n_1}$, $n=0,\ldots,n_1-1$ (cf. (7.2)).

If D_{n_1} is generated by $\mathbb{Z}_{n_1}$ as in (8.3) and by E as in $(7.2)_2$ then (8.1) implies (8.2).

If "a" is an eigenvalue of A with eigenspace V_{ℓ}, then (8.2) still holds under extra assumptions. Namely, if $Fix_{\mathbb{Z}_{n_1}\times\{I\}}(\mathcal{N}_0)=V_{\ell}\oplus U_{\ell_1,n_1}$ for some $n_1\in\mathbb{N}$, and if $Fix_{\Sigma}(V_{\ell})=\{0\}$ but $\dim Fix_{\Sigma}(V_{\ell_1})=d$, then $Fix_{D_{n_1}\times\Sigma}(\mathcal{N}_0)\subset U_{\ell_1,n_1}$ and (8.2) holds.

Since solutions with isotropy $D_{n_1}\times\Sigma$ have the spatial symmetries characterized by Σ for all times $t\in\mathbb{R}$ (mod P/n_1) we call these solutions *standing waves*.

Subgroups Σ of $O(3)$ having one-dimensional fixed-point subspace in some V_ℓ for the natural action (7.1) are of special interest. The reason for this is clear—we then obtain a bifurcating curve of solutions $\{(\mu(\varepsilon), u(\varepsilon))\}$ with $u(\varepsilon)(t, x) = \varepsilon\psi(t, x) + o(|\varepsilon|)$ where $Fix_{D_{n_1} \times \Sigma}(\mathcal{N}_0) = span\{\psi(t, x)\}$ (cf. (4.5)). The isotropy of ψ and that of the nonlinear wave $u(\varepsilon)$ are the same (cf. Section 7) and for small ε the pattern of $u(\varepsilon)$ is certainly close to that of ψ which is known explicitely (cf. our examples below).

Fortunately a classification of all maximal isotropy subgroups Σ of $O(3)$ satisfying (8.1) with $d = 1$ for some $\ell_1 \in \mathbb{N}_0$ is well known, cf. [5, p. 131].

We now give some explicit examples adopting the notation of [5]. For the function h we assume the invariance (7.3) with respect to the group Σ, respectively.

I. $\Sigma = O(2) \oplus \mathbb{Z}_2^c$

The elements in $Fix_\Sigma(H^2(S^2))$ are called axisymmetric. For convenience we fix one copy of $O(2)$ which leaves the x_3-axis in $\mathbb{R}^3$ fixed. $\mathbb{Z}_2^c = \{\pm I\}$ is the centralizer of $O(3)$.

For the eigenvalue $\lambda_4 = 20$ the eigenfunction of $A = -\Delta$ in $Fix_\Sigma(H^2(S^2))$ is given by $\varphi_{4,0}(x) = \frac{1}{8}(35 \cos^4 \alpha - 30 \cos^2 \alpha + 3)$ (cf. (5.2)) and choosing $a = a(0) = 4$ and $P = 2\pi$ we find $S_{4, 2\pi} = \{(4, \pm 4)\}$ and therefore (8.1) and (8.2) are satisfied with $d = 1$ and $n_1 = 4$. Thus we obtain the kernel $Fix_{D_4 \times \Sigma}(N(L - aI)) = span\{\varphi_{4,0}(x) \cos 4t\}$ which gives a local solution curve with $u(\varepsilon)(t, x) = \varepsilon\varphi_{4,0}(x) \cos 4t + o(|\varepsilon|)$ having isotropy $D_4 \times \Sigma$. In Fig. 1 the nodal set of $\varphi_{4,0}$ is depicted which is not necessarily maintained for $u(\varepsilon) \in Fix_\Gamma(H^{2,2}(Q_{2\pi}))$, $\Gamma = D_4 \times \Sigma$. Since $a = a(0)$ is an integer we can

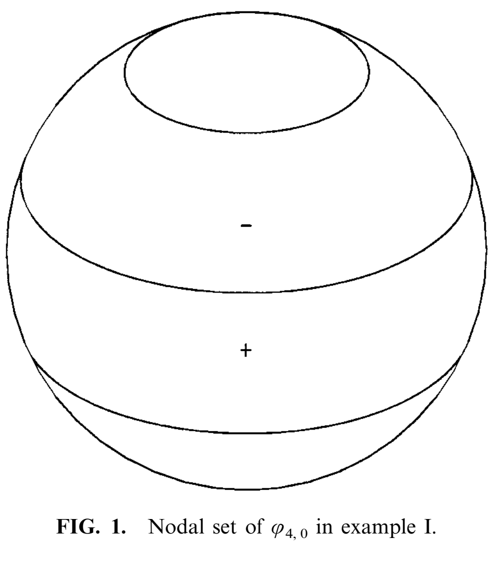

FIG. 1. Nodal set of $\varphi_{4,0}$ in example I.

apply also Theorem 7.3 and we get a global branch C_{P_4}, $P_4 = 2\pi/4$, of solutions for (5.10) in $\mathbb{R} \times Fix_\Gamma(H^{2,2}(Q_{2\pi}))$.

We also find axisymmetric solutions when "a" is an eigenvalue of $A = -\Delta$. This happens e.g. at $a = 2$. Here $S_{2,2\pi} = \{(1,0), (2, \pm 2)\}$ and from [5], Chapter XIII, Theorem 9.9, dim $Fix_\Sigma(V_1) = 0$ and dim $Fix_\Sigma(V_2) = 1$. Hence, we obtain a local smooth curve of $P_2 = 2\pi/2$-periodic solutions with isotropy $D_2 \times \Sigma$ which is different from the global branch of nontrivial stationary solutions bifurcating from (0, 0) as well. The curve of nonstationary solutions is globally extended, too, but it could meet the stationary branch.

The following subgroups $\Sigma \subset O(3)$ often involve the exceptional subgroups $\mathbb{T}, \mathbb{O}$, and $\mathbb{I}$ of $SO(3)$, called the tetrahedral, octahedral, and icosahedral subgroups, respectively. We now fix a copy of each of these groups. Concerning $\mathbb{T}$ we choose the following: The elements of order two send two variables to their respective negatives, and one element of order three gives a cyclic permutation of the three variables, cf. (10.1).

For $\mathbb{O} \subset O(3)$ we choose the unique subgroup, which is a supergroup of $\mathbb{T}$ as chosen above. We fix $\mathbb{I}$ to be the icosahedral subgroup which contains two $\mathbb{Z}_5$ subgroups which rotate around the axis (0, 0, 1) and $(-2/\sqrt{5}, 0, 1/\sqrt{5})$, respectively.

II. $\Sigma = \mathbb{O} \oplus \mathbb{Z}_2^c$

Choosing the same parameters $\lambda_4 = 20$, $a = a(0) = 4$, $P = 2\pi$, $n_1 = 4$, we get the same results as for Example I with the eigenfunction $\hat{\varphi}_4(x) = 168\varphi_{4,0}(x) + \varphi_{4,4}(x)$ (cf. (5.2)) in $Fix_\Sigma(H^2(S^2))$. Here $Fix_{D_4 \times \Sigma}(N(L - aI)) = span\{\hat{\varphi}_4(x) \cos 4t\}$. Its spatial nodal set is shown in Fig. 2.

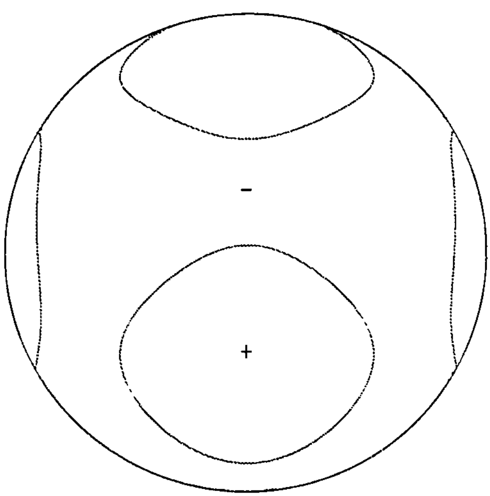

FIG. 2. Nodal set of $\hat{\varphi}_4$ in example II.

We also find bifurcation of nontrivial periodic solutions when "a" is an eigenvalue of A. E.g. let $a=6$, then $S_{6,\,2\pi}=\{(2,0),(6,\pm 6)\}$ and (cf. [5, Chapter XIII, Table 8.1]) $\dim Fix_{\Sigma}(V_2)=0$ and $\dim Fix_{\Sigma}(V_6)=1$. Therefore we get a $P_6=(2\pi/6)$-periodic solution curve with isotropy $D_6\times\Sigma$, which occurs in addition to a branch of stationary solutions. For the global extensions of both branches the comments of Example I hold here as well.

III. $\Sigma=\mathbb{I}\oplus\mathbb{Z}_2^c$

Here we choose $\lambda_6=42$, $a=a(0)=-7$, $P=2\pi$, giving $S_{-7,\,2\pi}=\{(1,\pm 3),(6,\pm 7)\}$ and (8.2) applies for $d=1$ and $n_1=7$. The eigenfunction in $Fix_{\Sigma}(H^2(S^2))$ is given by $\hat{\varphi}_6(x)=-3960\varphi_{6,\,0}(x)+\varphi_{6,\,5}(x)$ whose nodal set is sketched in Fig. 3. Since "a" is an integer, the local curve $\{(\mu(\varepsilon),u(\varepsilon))\}$, where $u(\varepsilon)(t,x)=\varepsilon\hat{\varphi}_6(x)\cos 7t+o(|\varepsilon|)$, is again globally extended in $\mathbb{R}\times Fix_{D_7\times\Sigma}(H^{2,\,2}(Q_{2\pi}))$ (cf. Theorem 7.3).

Similarly, we could choose $\lambda_6=42$, $a=a(0)=\frac{47}{4}$ and $P=P_{\min}=4\pi/11$ (see Remark 5.12). In this second example, however, "a" is not an integer, and therefore we cannot apply Theorem 6.3 or Theorem 7.3, i.e. we do not know whether the local curve with $u(\varepsilon)(t,x)=\varepsilon\hat{\varphi}_6(x)\cos\frac{11}{2}t+o(|\varepsilon|)$ is globally extended.

IV. $\Sigma=D_6^d$

As dihedral group $D_6^d\subset O(3)$ we choose that one with x_3-axis as axis of rotation. The eigenvalue $\lambda_3=12$ is then simple in $Fix_{\Sigma}(H^2(S^2))$ and choosing $a=a(0)=3$, $P=2\pi$, we find $S_{3,\,2\pi}=\{(3,\pm 3)\}$ giving a solution curve

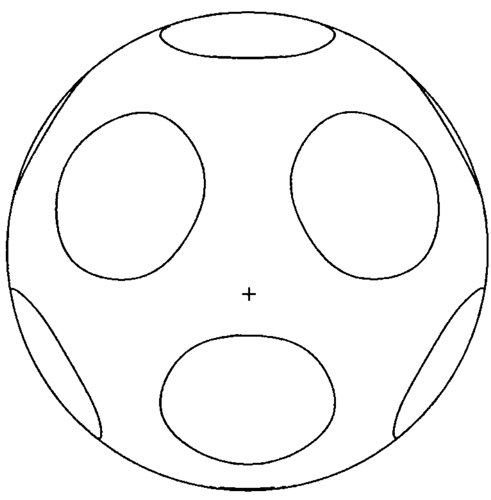

FIG. 3. Nodal set of $\hat{\varphi}_6$ in example III.

with period $P_3 = 2\pi/3$ and isotropy $D_3 \times D_6^d$. The eigenfunction $\varphi_{3,3}(x)$ (see (5.2)) is depicted in Fig. 4. The local curve is globally extended.

V. $\Sigma = \mathbb{O}$

Here we can choose the eigenvalue $\lambda_9 = 90$, $a = a(0) = 9$, $P = 2\pi$, such that $S_{9,2\pi} = \{(9, \pm 9)\}$ and a $P_9 = 2\pi/9$-periodic solution curve with isotropy $D_9 \times \Sigma$ bifurcates from the trivial solution. The local curve extends to a global continuum C_{P_9} in $\mathbb{R} \times Fix_\Gamma(H^{2,2}(Q_{2\pi}))$ with $\Gamma = D_9 \times \Sigma$. The eigenfunction of $A = -\Delta$ in $Fix_\Sigma(H^2(S^2))$ is given by $\hat{\varphi}_9(x) = -4080\varphi_{9,-4}(x) + \varphi_{9,-8}(x)$ (cf. (5.2)) whose nodal set is shown in Fig. 5.

VI. $\Sigma = \mathbb{O}^-$

We fix $\mathbb{O}^- = \mathbb{T} \cup \{-\gamma : \gamma \in \mathbb{O} \backslash \mathbb{T}\}$ with $\mathbb{T}$ being fixed as before.

The parameters $\lambda_3 = 12$, $a = a(0) = 3$, $P = 2\pi$, allow to apply (8.2) with $d = 1$ and $n_1 = 3$ and we obtain a $P_3 = (2\pi/3)$-periodic solution curve with isotropy $D_3 \times \Sigma$. The nodal lines of the eigenfunction $\varphi_{3,-2}(x)$ (see (5.2)) are depicted in Fig. 6. The local curve is globally extended.

VII. $\Sigma = \mathbb{I}$

The eigenvalue $\lambda_{15} = 240$ is simple in $Fix_\Sigma(H^2(S^2))$ and the choice of $a = a(0) = 15$, $P = 2\pi$, gives a period $P_{15} = 2\pi/15$ since $S_{15,2\pi} = \{(15, \pm 15)\}$. Since "$a$" is an integer Theorem 7.3 gives a global continuum. The eigenfunction $\hat{\varphi}_{15}(x) = -36306144000\varphi_{15,-5}(x) - 62640\varphi_{15,-10}(x) + \varphi_{15,-15}(x)$ is represented in Fig. 7.

The bifurcating nonlinear waves of the preceeding seven examples are locally of the form $u(\varepsilon)(t, x) = \varepsilon\hat{\varphi}_\ell(x)\cos(2\pi/P)\,nt + o(|\varepsilon|)$ (in the $H^{2,2}(Q_P)$-topology, where in all but one example we have $P = 2\pi$). Although, by

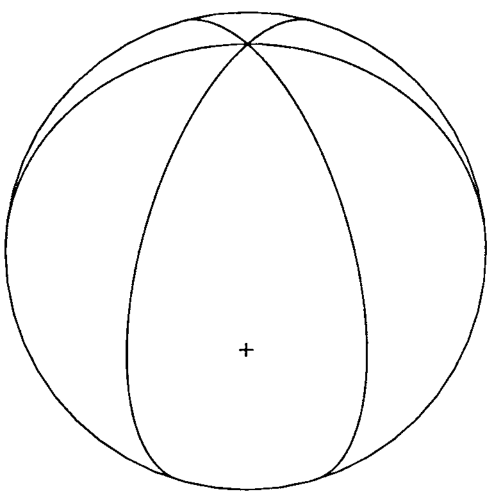

FIG. 4. Nodal set of $\varphi_{3,3}$ of example IV.

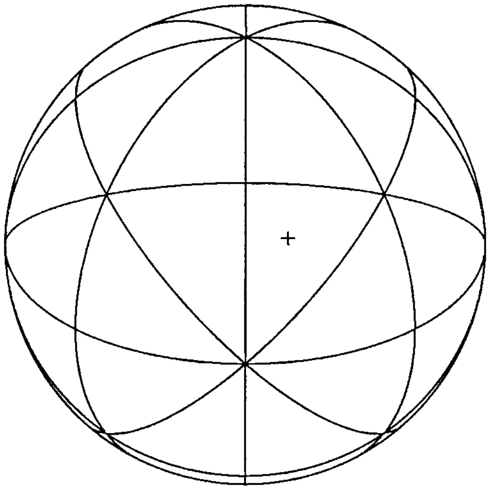

FIG. 5. Nodal set of $\hat{\varphi}_9$ of example V.

construction, $u(\varepsilon) \in Fix_{\Gamma}(H^{2,2}(Q_P))$ with isotropy $\Gamma = D_n \times \Sigma$, the spatial symmetry does not necessarily imply that the nodal curves of the eigenfunction $\hat{\varphi}_\ell$ as shown in Figs. 1–7 are preserved for $u(\varepsilon)$. In the presence of more symmetry, however, certain nodal families are, in fact, preserved along branches of solutions locally and globally.

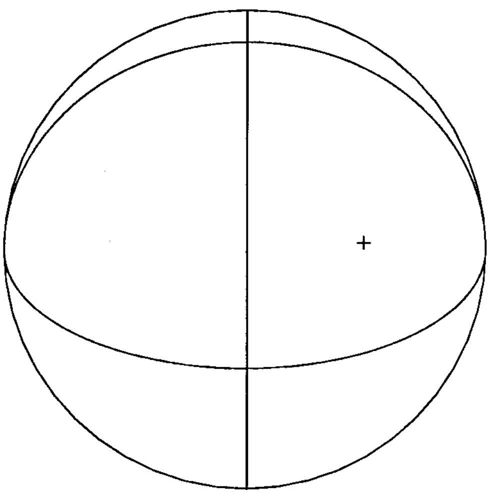

FIG. 6. Zero set of $\varphi_{3,-2}$ of example VI.

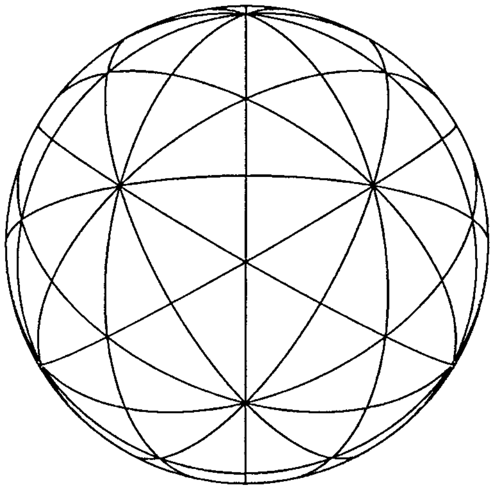

FIG. 7. Nodal set of $\hat{\varphi}_{15}$ of example VII.

In particular, in addition to (7.3), we assume

$$h(\mu, x, -u) = -h(\mu, x, u) \qquad \text{for all} \quad (\mu, x, u) \in \mathbb{R} \times S^2 \times \mathbb{R}. \tag{8.4}$$

We generate a group $\hat{\mathbb{Z}}_2$ by the action

$$(-\hat{I})\, u(t, x) = -u(t, x), \qquad (t, x) \in \mathbb{R} \ (\text{mod}\ P) \times S^2. \tag{8.5}$$

By assumptions (7.3) and (8.4) the mapping $G(\mu, \cdot)$ is $\Gamma \times \hat{\mathbb{Z}}_2$-equivariant in the sense of (7.4), where the action of $(\gamma, -\hat{I}) \in \Gamma \times \hat{\mathbb{Z}}_2$ on $u(t, x)$ is defined by $(\gamma, -\hat{I})\, u(t, x) = -(\gamma u(t, x))$, with $\gamma u(t, x)$ for $\gamma \in \Gamma = O(2) \times O(3)$ is given by (7.1) and (7.2).

Assume that

(8.6) $\sigma v = -v$ for all $v \in Fix_{\Sigma}(V_{\ell})$ and for some reflection $\sigma = \sigma_U \in O(3)$ across a plane U through the origin of $\mathbb{R}^3$.

We remark that we have $\sigma_U \notin \Sigma$, but usually $-\sigma_U \in \Sigma$. Let $\hat{\Sigma}$ be the subgroup of $O(3) \times \hat{\mathbb{Z}}_2$ generated by $\Sigma \times \{\hat{I}\}$ and the element $(\sigma_U, -\hat{I})$, where σ_U is given by (8.6). Then

$$Fix_{\hat{\Sigma}}(V_{\ell}) = Fix_{\Sigma}(V_{\ell}). \tag{8.7}$$

If dim $Fix_\Sigma(V_\ell)=1$ then λ_ℓ is a simple eigenvalue of $A=-\Delta$ in $Fix_\Sigma(H^2(S^2))$ and by (8.7) λ_ℓ is a simple eigenvalue in $Fix_{\hat\Sigma}(H^2(S^2))$. Restricting $G(\mu,\cdot)$ to $Fix_{\hat\Gamma}(H^{2,2}(Q_P))$ with $\hat\Gamma=D_n\times\hat\Sigma$ we can apply our local and global bifurcation theory for $G(\mu,u)=0$ as expounded before in this and the previous section.

By definition of $\hat\Gamma$ all functions in $Fix_{\hat\Gamma}(H^{2,2}(Q_P))$ are invariant under $(\sigma_U,-\hat I)$, in other words

$$\sigma_U u(t,x)=-u(t,x) \qquad \text{for all} \quad (t,x)\in\mathbb{R}(\text{mod } P)\times S^2. \tag{8.8}$$

Since $(\sigma_U,-\hat I)$ represents an "inverse reflection" of $u(t,\cdot)$ (i.e., a reflection followed by negation) across the plane U, this, in turn, implies that

(8.9) $U\cap S^2$ is a nodal curve of $u(t,\cdot)$ for all $t\in\mathbb{R}$ (mod P).

We summarize:

THEOREM 8.1. *If* dim $Fix_\Sigma(V_\ell))=1$ *with eigenfunction* $\hat\varphi_\ell\in Fix_\Sigma(V_\ell)$ *having a nodal curve* $U\cap S^2$ ($=$ *a great circle on* S^2) *such that property* (8.6) *holds, then that great circle is also a nodal curve for the nonlinear wave* $u(t,x)$ *solving* (5.10) *locally or globally in* $Fix_{\hat\Gamma}(H^{2,2}(Q_P))$, $\hat\Gamma=D_n\times\hat\Sigma$. *Since* Σ *is contained in* $\hat\Sigma$ *the other symmetries are also preserved for* $u(t,x)$.

When looking at our examples, we see that property (8.6) is fulfilled only for odd ℓ. Indeed, when σ_U satisfies (8.6) we obviously have $\sigma_U\notin\Sigma$, but the condition $\sigma_U v=-v$ on $Fix_\Sigma(V_\ell)$ is forced when $(-I)\circ\sigma_U\in\Sigma$ and $(-I)\in O(3)$ acts as minus identity on $Fix_\Sigma(V_\ell)$. Therefore (8.6) is satisfied for all odd ℓ and all reflections σ_U such that $-\sigma_U\in\Sigma$. For even ℓ we have $(-I)\in O(3)$ acting as the identity on $Fix_\Sigma(V_\ell)$, and this trick is not possible.

Only examples IV–VII allow an odd number ℓ. The nodal curves of the eigenfunctions in $Fix_\Sigma(V_\ell)$ for $\Sigma=D_6^d$, $\ell=3$, $\Sigma=\mathbb{O}$, $\ell=9$, $\Sigma=\mathbb{O}^-$, $\ell=3$, and $\Sigma=\mathbb{I}$, $\ell=15$ are great circles on S^2 and property (8.6) can be verified. That means that Figs. (4–7) also depict nodal circles of the nonlinear waves $u\in Fix_{\hat\Gamma}(H^{2,2}(Q_P))$ of (5.10) which exist globally according to Theorem 7.3 in each case. Since these nodal sets are frozen these solutions are (nonlinear) standing waves in a more restricted sense.

Remark 8.2. If a nodal curve of the corresponding eigenfunction is not a great circle, then it need not be preserved for the nonlinear wave solution. This is seen by the example $\Sigma=\mathbb{O}$, $\ell=13$: The nodal set of the eigenfunction in $Fix_\Sigma(V_\ell)$ is depicted in Fig. 8—only the great circles are preserved.

As pointed out before, property (8.6) can be verified only for odd ℓ. For even ℓ assumption (8.4) leads also to new phenomena which we describe now.

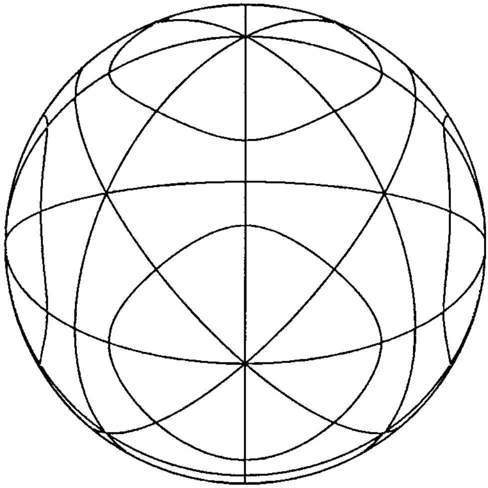

FIG. 8. Nodal set of an eigenfunction according to Remark 8.2.

The group $O(3)$ is decomposed into $SO(3)$ and $\mathbb{Z}_2^c$ generated by $-I \in O(3)$, i.e. $O(3) = SO(3) \oplus \mathbb{Z}_2^c$. For the natural action (7.1) $-I$ acts trivially on V_ℓ for even ℓ, but $-\hat{I}$ does not (cf. (8.5)). Therefore the action of $O(3) \times \hat{\mathbb{Z}}_2$ on V_ℓ for even ℓ is the same as the minus action of $SO(3) \oplus \mathbb{Z}_2^c \cong O(3)$ on V_ℓ. Fortunately for that action the subgroups of $O(3)$ having one-dimensional fixed-point subspaces are classified in [5, p. 129]. A new example is the following:

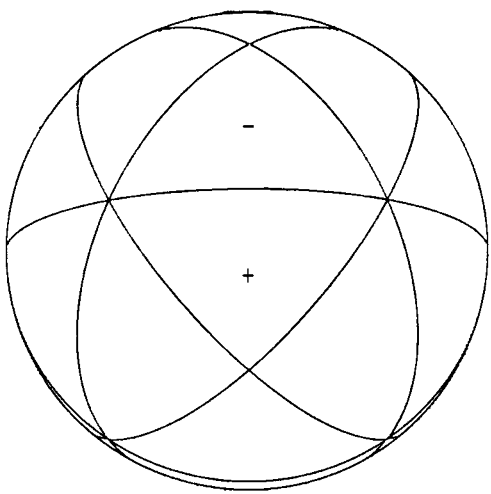

FIG. 9. Nodal set of $\check{\varphi}_6$ of example VIII.

VIII. $\Sigma = \mathbb{O}^- = \mathbb{T} \cup \{(\gamma, -\hat{I}): \gamma \in \mathbb{O} \backslash \mathbb{T}\} \subset SO(3) \times \hat{\mathbb{Z}}_2$

The eigenvalue $\lambda_6 = 42$ is simple in $Fix_\Sigma(H^2(S^2))$ and since $S_{-7, 2\pi} = \{(1, \pm 3), (6, \pm 7)\}$ we can choose $a = a(0) = -7$, $P = 2\pi$, and (8.2) applies with $d = 1$ and $n_1 = 7$. We obtain a curve of $P_7 = 2\pi/7$-periodic solutions. The eigenfunction in $Fix_\Sigma(V_6)$ is $\check{\varphi}_6(x) = -792\varphi_{6,2}(x) + \varphi_{6,6}(x)$ (cf. (5.2)) whose nodal lines are depicted in Fig. 9. The local curve is globally extended in $\mathbb{R} \times Fix_\Gamma(H^{2,2}(Q_{2\pi}))$, where $\Gamma = D_7 \times \Sigma$.

We see that the nodal curves of the eigenfunction $\check{\varphi}_6$ are great circles, which are nonlinearly preserved, because $-\sigma_U \in \mathbb{O}^-$, where U is a plane spanned by, e.g., e_1 and $e_2 + e_3$, where e_i are the unit vectors in $\mathbb{R}^3$.

Remark 8.3. The action of $O(3) \times \hat{\mathbb{Z}}_2$ (by (7.1) and (8.5)) on V_ℓ for odd ℓ is the same as the natural action of $SO(3) \times \hat{\mathbb{Z}}_2 \cong SO(3) \oplus \mathbb{Z}_2^c \cong O(3)$ and no new standing waves are found by exploiting the symmetry of $\hat{\mathbb{Z}}_2$.

9. ROTATING WAVES

Another class of solutions arises as a consequence of equivariance. In the next two sections, we consider "twisted" subgroups $\Gamma \subset O(2) \times O(3)$ for which the spatial and temporal group actions are coupled. In this section we focus on continuous twisted subgroups.

As in Section 8, the goal here is to find twisted subgroups Γ such that $\dim Fix_\Gamma(\mathcal{N}_0) = 1$. The group-theoretic results for continuous twisted subgroups of $S^1 \times O(3)$ are well known, cf. [5, p. 401] and [14, p. 286].

In particular, the only candidates for such twisted subgroups are $(SO(2) \oplus \mathbb{Z}_2^c)^{\sim k} \subset O(2) \times O(3)$, $k \subset \mathbb{N}$, which we now describe.

The group $SO(2) \subset O(3)$ is given by counter-clockwise rotations σ in the $x_1 x_2$-plane. The so-called twist $\theta_k: SO(2) \to S^1 \subset O(2)$ is defined as follows: If $\psi \in [0, 2\pi)$ denotes the angle of rotation of $\sigma \in SO(2)$, then for $k \in \mathbb{N}$,

$$\theta_k(\sigma) = T_{(k\psi/2\pi)(P/n_1)}, \tag{9.1}$$

where $T_\kappa \in O(2)$ is the time shift (mod P/n_1) as defined in (7.2).

(As in (8.3), we have allowed for a possible reduction of the admissible period P to P/n_1, where $n_1 \in \mathbb{N}$.) Then, by definition,

$$\Gamma_k^\sim = \{(\theta_k(\sigma), \sigma): \sigma \in SO(2)\} \subset O(2) \times O(3). \tag{9.2}$$

The (natural) action of $\Gamma_k^{\sim}$, as introduced in (7.1), (7.2), is then as follows:

$$(\theta_k(\sigma), \sigma)\, u(t, x) = u\left(t - \frac{k\psi}{2\pi}\frac{P}{n_1}, \sigma^{-1}x\right), \tag{9.3}$$

where

$$\sigma = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } \psi \in \mathbb{R}(\text{mod } 2\pi),\ (t, x) \in \mathbb{R}(\text{mod } P/n_1) \times S^2.$$

This corresponds to a clockwise spatial rotation through an angle ψ about the x_3-axis accompanied by a temporal phase shift of $(k\psi/2\pi)(P/n_1)$. In other words, functions in the fixed-point space of $\Gamma_k^{\sim}$ rotate rigidly counterclockwise about the x_3-axis in such a way that the spatial angle of rotation is $2\pi/k$ after one time-period P/n_1. This is, by definition, a *rotating wave.*

Remark 9.1. The nomenclature $(SO(2) \oplus \mathbb{Z}_2^c)^{\sim k}$ is adopted from [14]. In definition (9.2), however, the so-called centralizer $\mathbb{Z}_2^c$ does not occur. If we restrict the action (9.3) to U_{ℓ_1, n_1}, then in that (natural) representation, the group $\Gamma_k^{\sim}$ as defined by (9.2) is not an isotropy subgroup of $O(2) \times O(3)$. The twist θ_k can be extended to $SO(3) \oplus \mathbb{Z}_2^c$ such that the action on U_{ℓ_1, n_1} is the same as (9.3): if ℓ_1 is even then $\mathbb{Z}_2^c$ acts as the identity, if ℓ_1 is odd then a multiplication by -1 is equivalent to a time shift of half the period $P/2n_1$. The result of that extension is $(SO(2) \oplus \mathbb{Z}_2^c)^{\sim k}$, which is an isotropy subgroup of $O(2) \times O(3)$ acting on U_{ℓ_1, n_1}. However, the fixed-point subspaces of $(SO(2) \oplus \mathbb{Z}_2^c)^{\sim k}$ and of $\Gamma_k^{\sim}$ in U_{ℓ_1, n_1} are the same. Since a distinction between even and odd ℓ_1 does not make sense for an action on $H^{2,2}(Q_P)$ we retain definition (9.2).

As shown in [5, 14]

$$\dim Fix_{\Gamma_k^{\sim}}(U_{\ell_1, n_1}) = 2 \qquad \text{for} \quad k = 1, ..., \ell_1, \tag{9.4}$$

and it is easily verified that (in spherical coordinates)

$$Fix_{\Gamma_k^{\sim}}(U_{\ell_1, n_1}) = span\left\{\mathscr{P}_{\ell_1, k}(\cos\alpha)\cos\left(\frac{2\pi}{P}n_1 t - k\beta\right),\right.$$
$$\left.\mathscr{P}_{\ell_1, k}(\cos\alpha)\sin\left(\frac{2\pi}{P}n_1 t - k\beta\right)\right\} \tag{9.5}$$

(cf. (5.2)). In order to reduce the dimension to one we exploit more symmetry as follows: We define

(9.6) $\tilde{E}u(t, x) \equiv u(-t, x_1, -x_2, x_3)$ or in polar coordinates $\tilde{E}\tilde{u}(t, \alpha, \beta) = \tilde{u}(-t, \alpha, -\beta)$, $\alpha \in [0, \pi]$, $\beta \in [0, 2\pi)$, and $\tilde{\mathbb{Z}}_2 = \{I, \tilde{E}\}$.

Then, by the isotropy (9.6), the phase in $Fix_{\Gamma_k^{\sim}}(U_{\ell_1, n_1})$ is fixed and

(9.7) $Fix_{\Gamma_k^{\sim} \oplus \tilde{\mathbb{Z}}_2}(U_{\ell_1, n_1}) = span\{\mathscr{P}_{\ell_1, k}(\cos \alpha) \cos((2\pi/P)\, n_1 t - k\beta)\} = span\{\varphi_{\ell_1, k}(x) \cos(2\pi/P)\, n_1 t + \varphi_{\ell_1, -k}(x) \sin(2\pi/P)\, n_1 t\} \equiv span\{\hat{\psi}_{\ell_1, k, n_1}\}$.

Combining the groups defined by (9.2) and (9.6), viz., $\Gamma = \Gamma_k^{\sim} \oplus \tilde{\mathbb{Z}}_2$ we end up with

(9.8) $Fix_{\Gamma}(\mathscr{N}_0) = span\{\hat{\psi}_{\ell_1, k, n_1}\}$ where $\hat{\psi}_{\ell_1, k, n_1}$ is defined in (9.7) and where $\mathscr{N}_0 = N(L - aI)$ is given by (7.12)

(and "a" is not an eigenvalue of A). Therefore Theorem 7.3 yields a local curve of nonlinear rotating waves of (5.10) in $\mathbb{R} \times Fix_{\Gamma}(H^{2,2}(Q_P))$, which can be globally extended if $a = a(0)$ is an integer.

To be more precise, we need the equivariance of $G(\mu, \cdot)$ with respect to $\Gamma = \Gamma_k^{\sim} \oplus \tilde{\mathbb{Z}}_2$. According (7.3) we assume that

$$h(\mu, \sigma x, u) = h(\mu, x, u) \qquad \text{for all} \quad \sigma \in SO(2) \oplus \mathbb{Z}_2^c, \quad x \in S^2, \quad \mu, u \in \mathbb{R}. \tag{9.9}$$

Since $SO(2) \oplus \mathbb{Z}_2^c$ contains the spatial reflection symmetry of (9.6), Proposition 7.1 guarantees the Γ-equivariance and we can restrict $G(\mu, \cdot)$ to $Fix_{\Gamma}(H^{2,2}(Q_P))$. The local bifurcating curve is of the form $\{(\mu(\varepsilon), u(\varepsilon))\}$ where

$$u(\varepsilon)(t, x) = \varepsilon \hat{\psi}_{\ell_1, k, n_1}(t, x) + o(|\varepsilon|) \tag{9.10}$$

and $o(|\varepsilon|)$ is a perturbation in the $H^{2,2}(Q_P)$-topology. According to Section 7 it has the isotropy Γ. Of course we can create all local solutions $u(t, x)$ by arbitrary phase shifts in t. (The additional symmetry $\tilde{E}$ fixes the phase.) Whenever $a = a(0)$ is an integer Theorem 7.3 is applicable and the local curve (9.10) (together with all phase shifts) is extended globally in $\mathbb{R} \times Fix_{\Gamma}(H^{2,2}(Q_P))$.

We give two examples:

IX. $\Gamma_1^{\sim}$ or $(SO(2) \oplus \mathbb{Z}_2^c)^{\sim 1}$

For $a = a(0) = 3$, $\lambda_3 = 12$, and $P = 2\pi$, we obtain (9.8) with $n_1 = 3$, $\ell_1 = 3$, and $\hat{\psi}_{3,1,3}(t, x) = \varphi_{3,1}(x) \cos 3t + \varphi_{3,-1}(x) \sin 3t$, which is depicted in Fig. 10 for times $t = 0$, $\frac{1}{8} P_3$, $\frac{2}{8} P_3$, and $\frac{3}{8} P_3$ where $P_3 = 2\pi/3$. Since $k = 1$ the nonlinear wave rotates counter-clockwise about 2π during one period $2\pi/3$. Due to Theorems 6.3 and 7.3 the local curve is extended globally as a rotating wave.

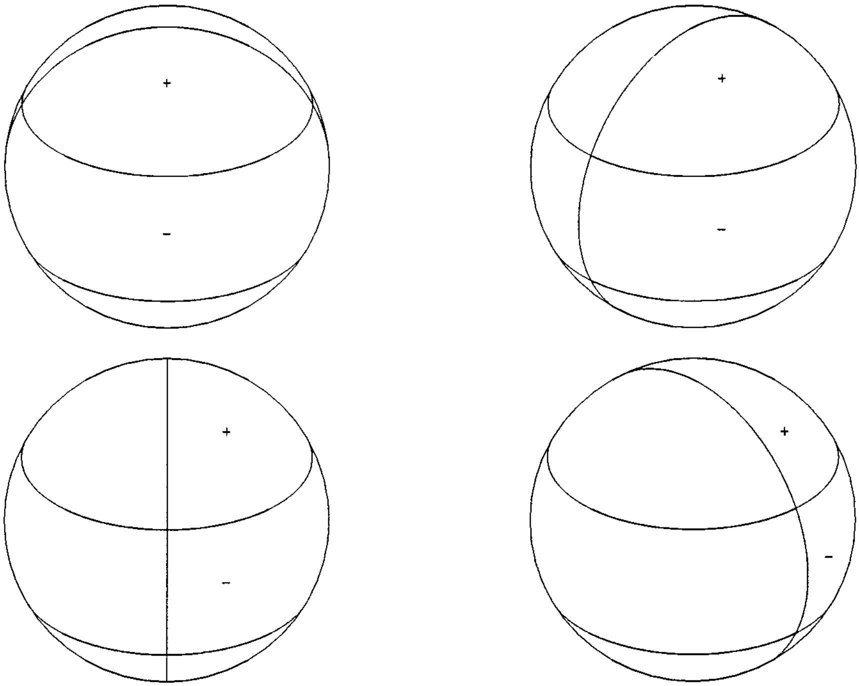

FIG. 10. Nodal set of $\hat{\psi}_{3,1,3}$ of example IX.

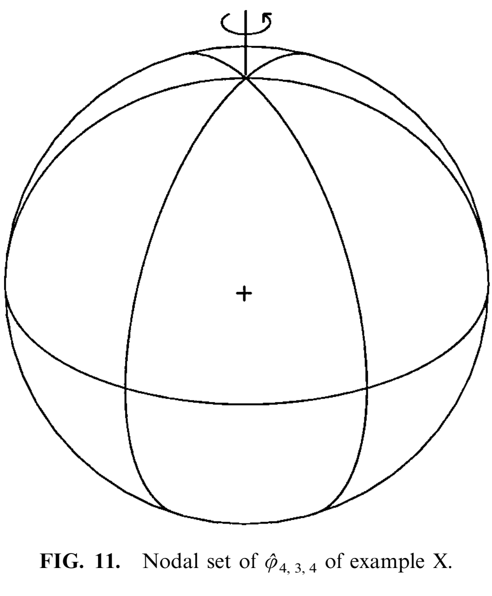

FIG. 11. Nodal set of $\hat{\varphi}_{4,3,4}$ of example X.

X. $\Gamma_3^{\sim}$ or $(SO(2) \oplus \mathbb{Z}_2^c)^{\sim 3}$

Here we choose $a = a(0) = 4$, $\lambda_4 = 20$, and $P = 2\pi$. Then (9.8) holds with $n_1 = 4$, $\ell_1 = 4$, and the eigenfunction $\hat{\psi}_{4,3,4}(t, x) = \varphi_{4,3}(x) \cos 4t + \varphi_{4,-3}(x) \sin 4t$ is shown in Fig. 11. It rotates counter-clockwise about 2π during one period $\pi/2$. Again the local curve is globally extended in $\mathbb{R} \times Fix_\Gamma(H^{2,2}(Q_{2\pi}))$.

Figures 10 and 11 show the nodal curves of the linear wave $\hat{\psi}_{\ell_1, k, n_1}$, respectively, which could be perturbed for the nonlinear wave (9.10) given by Theorems 4.2 or 7.3. Under the additional assumption (8.4), however, some typical nodal lines of $\hat{\psi}_{\ell_1, k, n_1}$ can be preserved for the nonlinear wave locally and globally.

For that purpose we replace (9.6) by

(9.11) $\check{E}u(t, x) = -u(-t, x_1, -x_2, x_3)$ or in spherical coordinates $\check{E}\tilde{u}(t, \alpha, \beta) = -u(-t, \alpha, -\beta)$, $\alpha \in [0, \pi]$, $\beta \in [0, 2\pi)$ and $\check{\mathbb{Z}}_2 = \{I, \check{E}\}$.

The actions (9.3) and (9.11), imply that $\tilde{u} \in Fix_{\Gamma_k^{\sim} \oplus \check{\mathbb{Z}}_2}(H^{2,2}(Q_P))$ satisfies (in spherical coordinates)

$$\begin{aligned} \tilde{u}(t, \alpha, \beta) &= \tilde{u}\left(t - \frac{k\psi}{2\pi} \frac{P}{n_1}, \alpha, \beta - \psi\right), \\ \tilde{u}(t, \alpha, \beta) &= -\tilde{u}(t, \alpha, -\beta). \end{aligned} \tag{9.12}$$

Combining both symmetries (9.12) with the periodicity $u(t + (P/n_1), x) = u(t, x)$ we obtain (in spherical coordinates)

$$\tilde{u}\left(t, \alpha, \frac{2\pi}{kP} n_1 t + \frac{m\pi}{k} + \tilde{\beta}\right) = -\tilde{u}\left(t, \alpha, \frac{2\pi}{kP} n_1 t + \frac{m\pi}{k} - \tilde{\beta}\right),$$
$$\text{for all} \quad t \in \mathbb{R}\left(\text{mod } \frac{P}{n_1}\right), \quad m \in \mathbb{Z}, \quad \tilde{\beta} \in \mathbb{R}. \tag{9.13}$$

In particular, the longitudinal great circles

(9.14) $(x_1, x_2, x_3) = (\sin\alpha \cos((2\pi/kP) n_1 t + (m\pi/k)), \sin\alpha \sin((2\pi/kP) n_1 t + (m\pi/k)), \cos\alpha)$ for all $\alpha \in [0, \pi]$, $t \in \mathbb{R}(\text{mod } P/n_1)$, $m = 0, ..., 2k-1$, are nodal curves on S^2 for all $u \in Fix_\Gamma(H^{2,2}(Q_P))$ where $\Gamma = \Gamma_k^{\sim} \oplus \check{\mathbb{Z}}_2$.

In accordance with (9.3) these nodal circles rotate rigidly about an angle $2\pi/k$ during one period P/n_1.

Assumptions (8.4) and (9.10) imply that $G(\mu, \cdot)$ is equivariant with respect to Γ as given in (9.14), and, we see by (9.5) that

$$
\begin{aligned}
&Fix_{\Gamma}(\mathcal{N}_0) = span\{\check{\psi}_{\ell_1, k, n_1}\} \text{ where} \\
&\check{\psi}_{\ell_1, k, n_1}(t, x) = \varphi_{\ell_1, k}(x) \sin \frac{2\pi}{P} n_1 t - \varphi_{\ell_1, -k}(x) \cos \frac{2\pi}{P} n_1 t.
\end{aligned} \tag{9.15}
$$

The local curve given by Theorem 7.3 is of the form $\{(\mu(\varepsilon), u(\varepsilon))\}$ where

$$u(\varepsilon)(t, x) = \varepsilon \check{\psi}_{\ell_1, k, n_1}(t, x) + o(|\varepsilon|) \tag{9.16}$$

is obtained from (9.10) by a phase shift about $-P/4n_1$. Since all local and global nonlinear waves given by Theorem 7.3 are in $Fix_{\Gamma}(H^{2,2}(Q_P))$ with Γ is as in (9.14), all nodal curves of $\check{\psi}_{\ell, k, n_1}$ which are longitudinal great circles are nonlinearly preserved (for (9.16)).

Finally, an inverse reflection across the $x_1 x_2$-plane also maintains the equator as a nodal circle for all bifurcating nonlinear waves if the eigenfunction $\check{\psi}_{\ell, k, n_1}$ has that isotropy. This is the case for Example X. Therefore Fig. 11 shows not only the eigenfunction $\check{\psi}_{4,3,4}$ (or $\hat{\psi}_{4,3,4}$) but it represents also typical nodal circles of nonlinear rotating waves obtained by our local or global bifurcation theory.

Remark 9.2. Rotating waves having spatial symmetries (which rotate) as shown in Figs. 10 and 11, e.g., cannot be solutions of an evolution equation which is of first order in t (e.g., a parabolic equation on the sphere). If a (periodic) solution has some spatial isotropy for any fixed time, then the fixed-point space of that isotropy subgroup Σ of $O(3)$, say, is invariant for all times "t". (Restrict the evolution equation to that fixed-point space and observe that the solution at some fixed time determines its evolution for all times in a unique way.) A fixed spatial isotropy Σ is compatible with the defining property (9.3) of a rotating wave only if $\Sigma = \mathbb{Z}_2^c$, $SO(2)$, or $SO(2) \oplus \mathbb{Z}_2^c$. In the cases when Σ contains $SO(2)$, however, a rotating wave is the same as an axisymmetric stationary wave. In this sense, genuine rotating waves for evolution equations which are of first order in t, can have, at most, the spatial isotropy $\mathbb{Z}_2^c$. A glance at our examples shows that rotating waves of a wave equation (which is of second order in t) can have many spatial symmetries (which rotate rigidly).

10. DISCRETE-ROTATING WAVES

For a rotating wave, defined in the previous section, the image of the twist is continuous, viz., $S^1 \subset O(2)$, cf. (9.1). If, however, we define the twist θ on a finite group $\Sigma \subset O(3)$ the image is also a finite subgroup $\mathbb{Z}_k$. Again, the group theoretic calculations have been carried out in [5, p. 401, and 14]. In particular, it is known that the only candidate for such a discrete twisted subgroup of $S^1 \times O(3)$ is $(\mathbb{T} \oplus \mathbb{Z}_2^c)^\sim$, which we now describe. Here $\mathbb{T}$ is the tetrahedral subgroup of $O(3)$, which is generated by

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.1}$$

which are elements of order 3 and 2, respectively. We define the twist $\theta\colon \mathbb{T} \to S^1 \subset O(2)$ (a homeomorphism) by

$$\theta(\sigma_1) = T_{P/3n_1} \qquad \text{and} \qquad \theta(\sigma_2) = T_0 \text{ (cf. (7.2))} \tag{10.2}$$

where we assume that we have reduced the period P to P/n_1, analogously to (8.3). Then clearly the image of the twist is $\{T_0, T_{P/3n_1}, T_{2P/3n_1}\} \cong \mathbb{Z}_3$ and the twisted subgroup is, by definition,

$$\Gamma^\sim = \{(\theta(\sigma), \sigma) \colon \sigma \in \mathbb{T}\}. \tag{10.3}$$

The natural action of $\Gamma^\sim$ is (cf. (7.1), (7.2))

$$(\theta(\sigma), \sigma)\, u(t, x) = u(t - \theta(\sigma), \sigma^{-1} x) \tag{10.4}$$

where $(t, x) \in \mathbb{R}\ (mod\ P/n_1) \times S^2$.
Functions in the fixed-point space of $\Gamma^\sim$ have the isotropy

$$\begin{aligned} u\left(t - \frac{P}{3n_1}, x\right) &= u(t, \sigma_1 x), \\ u(t, x) &= u(t, \sigma_2 x) \qquad \text{for} \quad (t, x) \in \mathbb{R}\left(\text{mod}\, \frac{P}{n_1}\right) \times S^2, \end{aligned} \tag{10.5}$$

which is described as follows: A time shift of length $P/3n_1$ (= one third of the period) is equivalent to a rotation of the spatial variable x about (1, 1, 1) by an angle $2\pi/3$. For all times "t" the functions have the spatial symmetry of a rotation about (0, 0, 1) (= the x_3-axis) through the angle π. These symmetries describe, by definition, a *discrete-rotating wave*. Note that, apart from the $\mathbb{Z}_2$-isotropy $(10.5)_2$ in the $x_1 x_2$-plane, functions in the fixed-point space of $\Gamma^\sim$ do not rotate rigidly: only after one-third of the period, the same pattern, rotated by $2\pi/3$, reappears.

A remark similar to Remark 9.1 holds here as well to explain the notation $(\mathbb{T} \oplus \mathbb{Z}_2^c)^\sim$, which is adopted from [14].

In particular, $\Gamma^\sim$ may be employed for the purposes of fixed-point subspaces of U_{ℓ_1, n_1}; in [5, 14] it is shown that

$$\dim Fix_{\Gamma^\sim}(U_{\ell_1, n_1}) = 2 \qquad \text{for} \quad \ell_1 = 2, 4, 5, 6, 7, 9. \tag{10.6}$$

We confine our subsequent calculations to $\ell_1 = 4$; the other cases are analogous. With the notation of (5.2) we obtain

$$Fix_{\Gamma^\sim}(U_{4,n_1}) = span\Big\{8\sqrt{3}\,\varphi_{4,2}(x)\cos\frac{2\pi}{P}n_1 t + (\varphi_{4,4}(x) - 120\varphi_{4,0}(x))\sin\frac{2\pi}{P}n_1 t,$$

$$8\sqrt{3}\,\varphi_{4,2}(x)\sin\frac{2\pi}{P}n_1 t - (\varphi_{4,4}(x) - 120\varphi_{4,0}(x))\cos\frac{2\pi}{P}n_1 t\Big\}. \tag{10.7}$$

With the following additional symmetry we reduce the dimension to one. We define

(10.8) $\bar{E}u(t, x) = u(-t, x_2, x_1, x_3)$ or in polar coordinates $\bar{E}\tilde{u}(t, \alpha, \beta) = \tilde{u}(-t, \alpha, (\pi/2) - \beta)$, $\alpha \in [0, \pi]$, $\beta[0, 2\pi)$, and $\bar{\mathbb{Z}}_2 = \{I, \bar{E}\}$.

Then

$$Fix_{\Gamma^\sim \oplus \bar{\mathbb{Z}}_2}(U_{4,n_1}) = span\Big\{8\sqrt{3}\,\varphi_{4,2}(x)\sin\frac{2\pi}{P}n_1 t - (\varphi_{4,4}(x) - 120\varphi_{4,0}(x))\cos\frac{2\pi}{P}n_1 t\Big\}$$

$$\equiv span\{\bar{\psi}_{4,n_1}\}. \tag{10.9}$$

Finally, combining the groups $\Gamma^\sim$ (cf. (10.3)), and $\bar{\mathbb{Z}}_2$ (cf. (10.8)) to $\Gamma = \Gamma^\sim \oplus \bar{\mathbb{Z}}_2$ we end up with

(10.10) $Fix_\Gamma(\mathcal{N}_0) = span\{\bar{\psi}_{4,n_1}\}$ where $\bar{\psi}_{4,n_1}$ is defined in (10.9) and $\mathcal{N}_0 = N(L - aI)$ is given by (7.12).

(We assume that "a" is not an eigenvalue of A.) If h has the invariance

(10.11) $h(\mu, \sigma x, u) = h(\mu, x, u)$ or all $\sigma \in \mathbb{T} \oplus \mathbb{Z}_2^c \oplus \mathbb{Z}_2^d$, $x \in S^2$, $\mu, u \in \mathbb{R}$, where $\mathbb{Z}_2^d$ is generated by the spatial reflection of (10.8),

then $G(\mu, \cdot)$ has the Γ-equivariance for $\Gamma = \hat{\Gamma} \oplus \Gamma^\sim \oplus \bar{\mathbb{Z}}_2$. Thus we can restrict $G(\mu, \cdot)$ to $Fix_\Gamma(H^{2,2}(Q_P))$ and an application of Theorems 4.2, 6.3, or 7.3 gives a local curve of nonlinear discrete-rotating waves of (5.10) in $\mathbb{R} \times Fix_\Gamma(H^{2,2}(Q_P))$ which is globally extended if $a = a(0)$ is an integer. The local curve is of the form $\{(\mu(\varepsilon), u(\varepsilon))\}$ where

$$u(\varepsilon)(t, x) = \varepsilon\bar{\psi}_{4,n_1}(t, x) + o(|\varepsilon|). \tag{10.12}$$

According to Section 7 it has the isotropy Γ. We give an example:

XI. $\Gamma^{\sim}$ or $(\mathbb{T} \oplus \mathbb{Z}_2^c)^{\sim}$

We choose $a = a(0) = 4$, $\lambda_4 = 20$, and $P = 2\pi$. Then (10.10) holds with $n_1 = 4$ and the eigenfunction $\bar{\psi}_{4,4}$ is given by (10.9) with $n_1 = 4$. The zero set of that function is shown in Fig. 12 at times $t = (k/24)\, P_4$, $k = 0, ..., 11$, where $P_4 = \pi/2$. The local curve is globally extended.

Remark 10.1. In Fig. 12 one observes more symmetry than required by the isotropy (10.5): after four time steps, i.e. after one sixth of the period, the same pattern reappears rotated about (1, 1, 1) by an angle $4\pi/3$ and multiplied by -1 (cf. (10.14) below). The reason for that additional symmetry is that $\bar{\psi}_{4,4}$ is in the fixed-point space of $(\mathbb{T} \times \hat{\mathbb{Z}}_2)^{\sim}$ defined as follows (for $\hat{\mathbb{Z}}_2$ see (8.5)). The twist θ: $\mathbb{T} \times \hat{\mathbb{Z}}_2 \to S^1$ is given by

$$\begin{aligned} &\theta(\sigma_1, \hat{I}) = T_{P/3n_1}, \qquad \theta(\sigma_1, -\hat{I}) = T_{5P/6n_1}, \\ &\theta(\sigma_2, \pm\hat{I}) = T_0. \end{aligned} \tag{10.13}$$

Then $(\mathbb{T} \times \hat{\mathbb{Z}}_2)^{\sim} = \{(\theta(\sigma), \sigma) \colon \sigma \in \mathbb{T} \times \hat{\mathbb{Z}}_2\}$, and the image of the twist is $\mathbb{Z}_6$.

If the function h has only the invariance (10.11), then the nonlinear wave (10.12) of (5.10) has only the isotropy (10.5).

However, if h satisfies in addition assumption (8.4), then $G(\mu, \cdot)$ is also equivariant with respect to the twisted subgroup $(\mathbb{T} \times \hat{\mathbb{Z}}_2)^{\sim}$. This means that the nonlinear wave (10.12) of (5.10) has locally (and globally) the isotropies (10.5), and in addition

$$u\left(t - \frac{P}{6n_1}, x\right) = -u\left(t, \sigma_1^2 x\right), \qquad \text{for} \quad (t, x) \in \mathbb{R}\left(\operatorname{mod} \frac{P}{n_1}\right) \times S^2. \tag{10.14}$$

Remark 10.2. By Theorem 4.1 or 7.3, we can prove bifurcation for any dimension of $Fix_{\Gamma}(\mathcal{N}_0)$. Therefore we do not need the restriction for ℓ_1 given in (10.6). For example, choosing the parameters $a = a(0) = 10$, $\lambda_{10} = 110$, and $P = 2\pi$, then, according to (10.6), dim $Fix_{\Gamma^{\sim}}(U_{10,10}) > 2$ and also $Fix_{\Gamma}(\mathcal{N}_0) > 1$. Nonetheless we get nontrivial nonlinear discrete-rotating waves for (5.10) in $\mathbb{R} \times Fix_{\Gamma}(H^{2,2}(Q_P))$ with period $P_{10} = 2\pi/10$.

Remark 10.3. Examples I, II, X, and XI are given for the parameters $a = 4$ and $P = 2\pi$. When applied to Example 6.6 (with that slight change of notation) $a = 4$ is a bifurcation point for (6.9), $P = 2\pi$, of a global solution continuum (cf. (6.12)). This global continuum contains two global continua of standing waves (Example I, II), one global continuum of rotating waves (Example X), and a global continuum of discrete-rotating waves (Example XI). Any two of these continua may meet only in the intersection of the

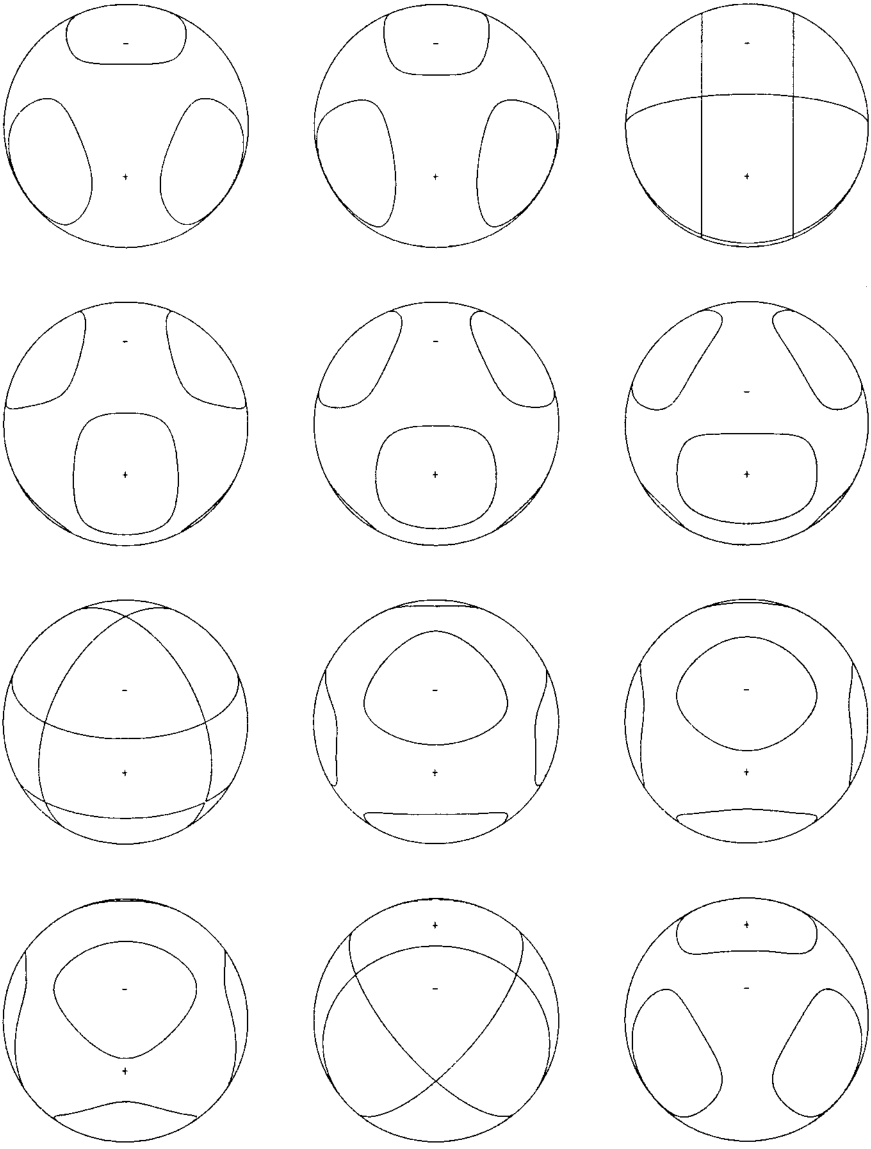

FIG. 12. Nodal set of $\bar{\psi}_{4,4}$ of example XI.

corresponding fixed-point subspaces of $\mathbb{R} \times H^{2,2}(Q_{2\pi})$. For instance, the axisymmetric standing wave of Example I and the rotating wave of Example X could meet at stationary axisymmetric solutions.

The same considerations hold for $a = 3$, $P = 2\pi$, in the Examples IV, VI, and IX, and for $a = -7$, $P = 2\pi$, in Examples III, VIII.

11. CONCLUDING REMARKS

It is remarkable that we are able to get global bifurcation results for (5.10) but not for (5.12). The basic reason for this is clear: in contrast to problem (5.10) with $a \in \mathbb{Z}$, the inverse operator (1.3) does not regularize (is not compact), in general, for (5.12). The underlying reason for this comes from an innocuous difference in the characteristic equations, cf. (2.13), (5.1). In problem (5.10) the appearance of the linear term "ℓ" from the eigenvalue (cf. (5.1)) in (2.13) plays a crucial role in the proof of compactness. This allows us to combine the finite-dimensional kernel (1.2) with the compactness of the inverse operator (1.3), cf. (6.1), (6.4). On the other hand, the eigenvalue $(\ell(\ell+1))^2$ appears in (2.13) for problem (5.12), and the characteristic equation now involves merely a sum of squares of integers, as in [6, 8]; from the former, we know that the inverse operator does not regularize unless the kernel (1.2) is infinite-dimensional, i.e. if $a = 0$. In this sense the results for (5.12) are in keeping with our past results for one-dimensional wave equations and for plate equations on two-dimensional domains.

Finally, we mention that the general results of Sections 2–6 can be extended to the sphere S^n for $n \geqslant 3$. The eigenvalues of the Laplace–Beltrami operator are given by $\lambda_\ell = \ell(\ell + n - 1)$ and a modification of the results of Sections 5 and 6 (including global bifurcation) is obvious. An exploitation of symmetry would also follow along the lines of Sections 7–10, but the group theory for $O(n)$ for $n \geqslant 4$ is much more involved. In particular, a classification of isotropy subgroups of $O(2) \times O(n)$ yielding one- or two-dimensional fixed-point spaces in the kernels $\mathcal{N}_0 = N(L - aI)$ would need to be carried out.

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