NONLINEAR STANDING WAVES IN ACOUSTIC RESONATORS WITH AN ARBITRARY REFLECTION COEFFICIENTS AT THEIR WALLS

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Abstract

This paper deals with the problems of description of finite-amplitude standing waves in gas-filled acoustic resonators, where the frontal resonator walls are of arbitrary reflection coefficients. The concrete wall-reflection coefficients form may be connected with the finite wall stiffness, radiation of acoustic energy from the resonator cavity or utilization of selective absorbing materials for nonlinear effects suppression. It is assumed that the standing waves are driven by vibrating piston or by external volume force, with arbitrary frequency of driving signal. One-dimensional model equation of the second order is used to derive a set of two non-homogenous Burgers equations describing two contrapropagating waves, which are connected at the resonator walls by the boundary conditions considered. The equations are solved numerically in frequency domain.

Introduction

Provided that acoustic standing wave is driven into high amplitude, the nonlinear effects are possible to distort originally harmonic wave and transform thus acoustic energy into higher harmonic components which result in heightened dissipation of acoustic energy. There are many methods of these nonlinear effects suppression such as appropriate shape of the resonant cavity and multifrequency driving signal, see [1], [2].

Other possibility is utilization of suitable materials at the resonator walls which absorb energy of the higher harmonic components or cause complying with the resonant condition only for the fundamental harmonics. It is necessary to derive suitable model equation supplemented with the appropriate boundary conditions to study behaviour of the nonlinear standing waves on these conditions.

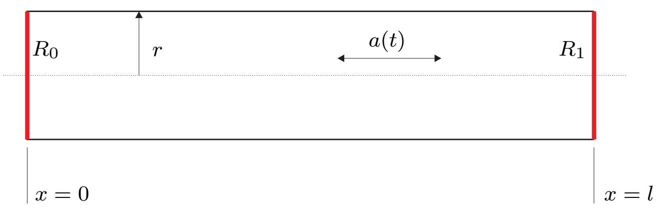

Figure 1: Acoustic resonator.

Model equations

The nonlinear standing waves in a cylindrical resonator, see figure 1, driven by means of external force may be described by an one-dimensional model equation in the second approximation, see [6], which has for constant radius form

$$\frac{\partial^2\varphi}{\partial t^2} - c_0^2\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial x}\right)^2 = -x\frac{\mathrm{d}a}{\mathrm{d}t} - a\frac{\partial\varphi}{\partial x} - \frac{\gamma-1}{2c_0^2}\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t} + ax\right)^2 + b\frac{\partial^3\varphi}{\partial x^2\partial t} - 2d\left(\frac{\partial^{3/2}\varphi}{\partial t^{3/2}} + x\frac{\mathrm{d}^{1/2}a}{\mathrm{d}t^{1/2}}\right), \quad (1)$$

where φ is a velocity potential, $a(t)$ is driving acceleration, x is spatial coordinate along the resonator body, t is time, c_0 is a small-signal sound speed, γ is the ratio of specific heats. Here b is the diffusity coefficient and d is a coefficient including influence of the boundary layer

$$b = \frac{1}{\rho_0}\left[\left(\zeta + \frac{4}{3}\eta\right) + \kappa\left(\frac{1}{c_V} - \frac{1}{c_p}\right)\right], \quad (2a)$$

$$d = \frac{\sqrt{\nu_0}}{r}\left(1 + \frac{\gamma-1}{\sqrt{Pr}}\right), \quad (2b)$$

where ρ_0 is ambient density, η and ζ are shear and bulk viscosities, κ is thermal conductivity coefficient, c_p and c_V are constant pressure and volume specific heats, $\nu_0 = \eta/\rho_0$ is kinematic viscosity, r is the resonator radius and Pr is the Prandtl number.

The half-order derivative in equation (1) is defined as

$$\frac{\partial^{\frac{1}{2}} f}{\partial t^{\frac{1}{2}}} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{\partial f(t')}{\partial t'}\frac{\mathrm{d}t'}{\sqrt{t-t'}}. \quad (3)$$

Equation (1) is written in coordinates moving with the resonator body; acoustic velocity may be obtained using its solution from definition of the velocity potential as

$$v = \frac{\partial\varphi}{\partial x}, \quad (4)$$

acoustic pressure $p' = p - p_0$ as

$$p' = -\rho_0\frac{\partial\varphi}{\partial t} - \rho_0 a\, x + \frac{\rho_0}{2c_0^2}\left(\frac{\partial\varphi}{\partial t} + ax\right)^2 - \frac{\rho_0}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{c_0^2}\left(\zeta + \frac{4}{3}\eta\right)\left(\frac{\partial^2\varphi}{\partial t^2} + x\frac{\mathrm{d}a}{\mathrm{d}t}\right). \quad (5)$$

This equation in the second approximation is derived from the Navier-Stokes equation.

If we assume the velocity potential as two counter-propagating waves, see [3],

$$\varphi(x,t) = \varphi_+(t_1 = \mu t, x_1 = \mu x, \tau_+ = t - x/c_0) + \\ + \varphi_-(t_1 = \mu t, x_1 = \mu x, \tau_- = t + x/c_0), \quad (6)$$

where μ is a small dimensionsless parameter, see [3], [4], and the driving acceleration as

$$a = a(t) = a_+(x,\tau_+) + a_-(x,\tau_-) \approx \mu^2, \quad (7)$$

$$a(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{j}k\omega t} = \quad (8)$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} \frac{a_k}{2} \mathrm{e}^{\mathrm{j}k\omega\tau_+} \mathrm{e}^{\mathrm{j}k\omega\frac{x}{c_0}}}_{a_+(x,\tau_+)} +$$

$$+ \underbrace{\sum_{k=-\infty}^{\infty} \frac{a_k}{2} \mathrm{e}^{\mathrm{j}k\omega\tau_-} \mathrm{e}^{-\mathrm{j}k\omega\frac{x}{c_0}}}_{a_- x,\tau_-}, \quad (9)$$

After neglecting terms of the order μ^3 and higher we obtain a set of equations

$$-2\mu c_0 \frac{\partial v_+}{\partial t_1} - 2\mu c_0^2 \frac{\partial v_+}{\partial x_1} + (\gamma+1) v_+ \frac{\partial v_+}{\partial \tau_+} = \\ = -x\frac{\partial a_+}{\partial \tau_+} - \frac{b}{c_0}\frac{\partial^2 v_+}{\partial \tau_+^2} + 2dc_0 \frac{\partial^{1/2} v_+}{\partial \tau_+^{1/2}}, \quad (10a)$$

$$2\mu c_0 \frac{\partial v_-}{\partial t_1} - 2\mu c_0^2 \frac{\partial v_-}{\partial x_1} + (\gamma+1) v_- \frac{\partial v_-}{\partial \tau_-} = \\ = -x\frac{\partial a_-}{\partial \tau_-} + \frac{b}{c_0}\frac{\partial^2 v_-}{\partial \tau_-^2} - 2dc_0 \frac{\partial^{1/2} v_-}{\partial \tau_-^{1/2}}. \quad (10b)$$

Here

$$v = \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi_+}{\partial x} + \frac{\partial \varphi_-}{\partial x} = v_+ + v_-. \quad (11)$$

In case of the steady-state conditions, the terms with the time derivatives can be neglected and set (10) is reduced to set of equations

$$\frac{\partial v_+}{\partial x} - \frac{\gamma+1}{2c_0^2} v_+ \frac{\partial v_+}{\partial \tau_+} = \quad (12a) \\ = \frac{x}{2c_0^2}\frac{\partial a_+}{\partial \tau_+} + \frac{b}{2c_0^3}\frac{\partial^2 v_+}{\partial \tau_+^2} - \frac{d}{c_0}\frac{\partial^{1/2} v_+}{\partial \tau_+^{1/2}},$$

$$\frac{\partial v_-}{\partial x} - \frac{\gamma+1}{2c_0^2} v_- \frac{\partial v_-}{\partial \tau_-} = \quad (12b) \\ = \frac{x}{2c_0^2}\frac{\partial a_-}{\partial \tau_-} - \frac{b}{2c_0^3}\frac{\partial^2 v_-}{\partial \tau_-^2} + \frac{d}{c_0}\frac{\partial^{1/2} v_-}{\partial \tau_-^{1/2}},$$

or written in a compact form

$$\frac{\partial v_\pm}{\partial x} - \frac{\gamma+1}{2c_0^2} v_\pm \frac{\partial v_\pm}{\partial \tau_\pm} = \\ = \frac{x}{2c_0^2}\frac{\partial a_\pm}{\partial \tau_\pm} \pm \frac{b}{2c_0^3}\frac{\partial^2 v_\pm}{\partial \tau_\pm^2} \mp \frac{d}{c_0}\frac{\partial^{1/2} v_\pm}{\partial \tau_\pm^{1/2}}. \quad (13)$$

For simple waves in nondissipative fluids we can write, see [5],

$$\frac{p_\pm}{p_0} = \left[\frac{c_0 \pm (\gamma-1)v_\pm/2}{c_0}\right]^{2\gamma/(\gamma-1)} \quad (14)$$

and after neglecting the high-order terms

$$p'_\pm \approx \pm\rho_0 c_0 v_\pm + \rho_0 \frac{\gamma+1}{4} v_\pm^2 + \ldots. \quad (15)$$

Acoustic pressure may be computed from solution of equation (13) using the first term of the equation (15).

Numerical analysis

For purposes of numerical analysis it is convenient to introduce non-dimensional variables

$$X = \frac{x}{l}, \quad T_\pm = \omega\tau_\pm, \quad V^\pm = \frac{v_\pm}{\pi c_0}, \\ P'^\pm = \frac{p'_\pm}{\pi^2 \rho_0 c_0^2}, \quad A^\pm = \frac{a_\pm}{l\omega_0^2}, \quad (16)$$

where

$$\omega_0 = \frac{\pi c_0}{l}$$

is the fundamental resonant frequency for rigid-walled resonator cavity and l is the resonator cavity length.

Non-dimensional form of equation (13) is

$$\frac{\partial V^\pm}{\partial X} = \pi^2 \frac{\gamma+1}{2}\Omega V^\pm \frac{\partial V^\pm}{\partial T_\pm} + \pi^2\Omega\frac{X}{2}\frac{\partial A^\pm}{\partial T_\pm} \pm \\ \pm \pi G_{TV}\frac{\Omega}{2}\frac{\partial^2 V^\pm}{\partial T_\pm^2} \mp \pi\Omega D \frac{\partial^{1/2} V^\pm}{\partial T_\pm^{1/2}}, \quad (17)$$

where

$$\Omega = \frac{\omega}{\omega_0}, \qquad G_{TV} = \frac{b\omega}{c_0^2}, \qquad D = \frac{d}{\sqrt{\omega}}.$$

For non-dimensional acoustic pressure we then obtain

$$P'^\pm = \pm\frac{V^\pm}{\pi}. \quad (18)$$

The driving is assumed to be periodic, so acoustic quantities can be expanded into Fourier series

$$V^\pm \approx \sum_{k=-N}^{N} V_k^\pm \mathrm{e}^{\mathrm{j}kT_\pm}, \quad (19a)$$

$$A^\pm \approx \sum_{k=-N}^{N} \frac{A_k}{2} \mathrm{e}^{\mathrm{j}kT_\pm} \mathrm{e}^{\pm\mathrm{j}k\pi\Omega X}. \quad (19b)$$

After substituting the series (19) into set (17), we obtain

$$\frac{\mathrm{d}V_k^\pm}{\mathrm{d}X} = \mathrm{j}\pi^2\frac{\gamma+1}{2}\Omega \sum_{m=k-N}^{N}(k-m)V_m^\pm V_{k-m}^\pm + \\ + \frac{\mathrm{j}k\pi^2\Omega X A_k}{4}\mathrm{e}^{\pm \mathrm{j}k\pi\Omega X} \mp \frac{\pi k^2 G_{TV}\Omega}{2}V_k^\pm \mp \\ \mp \pi\sqrt{\frac{|k|}{2}}\Omega D[1+\mathrm{j}\cdot\mathrm{sign}(k)]V_k^\pm. \quad (20)$$

If we assume the acoustic pressure reflection coefficients, see [5], at $X=0$ and $X=1$ to be $R_0 = R_0(\omega)$ and $R_1 = R_1(\omega)$, with respect to (11), we can write boundary conditions for acoustic velocity in form

$$X = 0: \quad (21a)$$
$$R_0 V^+ + V^- = 0 \;\Rightarrow\; R_0 V_k^+ + V_k^- = 0,$$
$$X = 1: \quad (21b)$$
$$V^+ + R_1 V^- = 0 \;\Rightarrow\; V_k^+ + R_1 V_k^- \mathrm{e}^{2\mathrm{j}k\pi\Omega X} = 0.$$

Set of ODEs (20) is solved numerically by means of the Runge-Kutta method of the 4^{th} order, the two-point boundary value problem (21) is solved using the shooting method.

Non-dimensional acoustic velocity spectra components may be expressed as

$$V_k = V_k^+\mathrm{e}^{-\mathrm{j}k\pi\Omega X} + V_k^-\mathrm{e}^{\mathrm{j}k\pi\Omega X} \quad (22)$$

and acoustic velocity time-domain dependence then

$$v(X,T) = 2\pi c_0 \sum_{k=1}^{N}\Big\{\big[\Re(V_k^+ + V_k^-)\cos k\pi\Omega X + \\ + \Im(V_k^+ - V_k^-)\sin k\pi\Omega X\big]\cos kT - \\ - \big[\Im(V_k^+ + V_k^-)\cos k\pi\Omega X - \\ - \Re(V_k^+ - V_k^-)\sin k\pi\Omega X\big]\sin kT\Big\}. \quad (23)$$

Numerical results

The following figures present some numerical results. For all cases the resonator cavity is filled with room-conditions air, the resonator length $l = 15$ cm, non-dimensional acceleration $A = 5\times10^{-4}$, $G_{TV} = 10^{-2}$, $D = 0$.

The figure 2 shows the comparison of numerical results obtained from Eqs. (13)+(15), and (1)+(5). Here $\Omega = 1$ and $R_0 = R_1 = 1$. The agreement of the numerical results obtained is evident, the algorithm for Eq. (13) numerical solution is faster, more stable and it is possible to process more harmonic components. It is also unnecessary to use so high additional-viscosity coefficient G_{TV}, see [1].

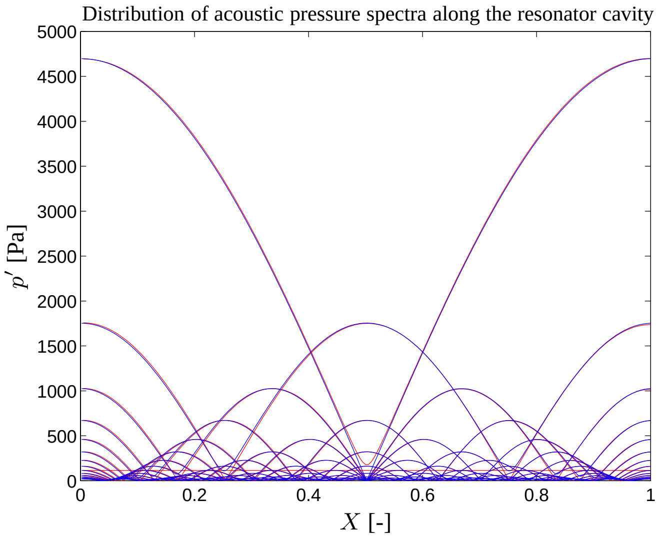

Figure 2: Comparison of numerical results. Red lines belong to solution of Eqs. (13)+(15), the blue lines to Eqs. (1)+(5).

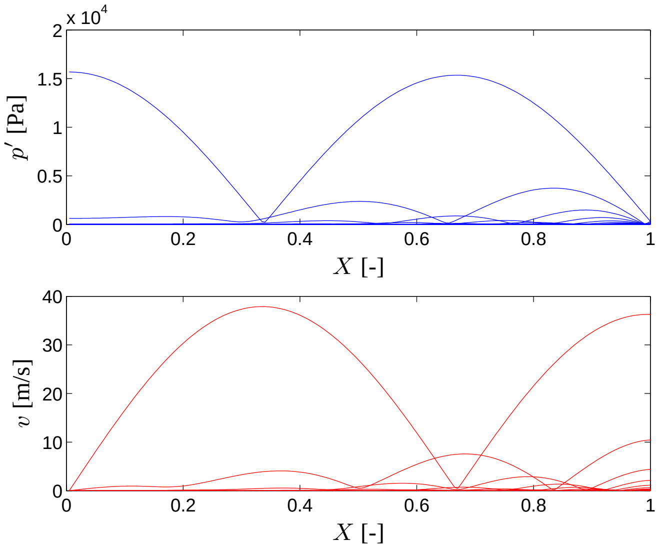

Figure 3: Distribution of acoustic pressure and velocity spectra for $R_0 = 1$, $R_1 = -1$.

The figure 3 shows the acoustic pressure and velocity spectra distribution in case of $R_0 = 1$, $R_1 = -1$ and $\Omega = 1.5$ (resonance condition for the fundamental harmonics). It can be seen that higher harmonics are suppressed and the first harmonic component is of high amplitude in comparison with the case of the figure 2. This can be explained by effect of the “time-reversal reflection”, see [7]. Wave traveling towards the wall with the reflection coefficient $R_1 = -1$ is distorted due to acoustic nonlinearities and it goes back to non-distorted form after the reflection.

The figure 4 shows the model case where all the energy of the higher harmonics is absorbed by the resonator boundaries. Here $\Omega = 1$, $R_0 = R_1 = 1$ for the fundamental harmonics and $R_0 = R_1 = 0$ for the higher ones.

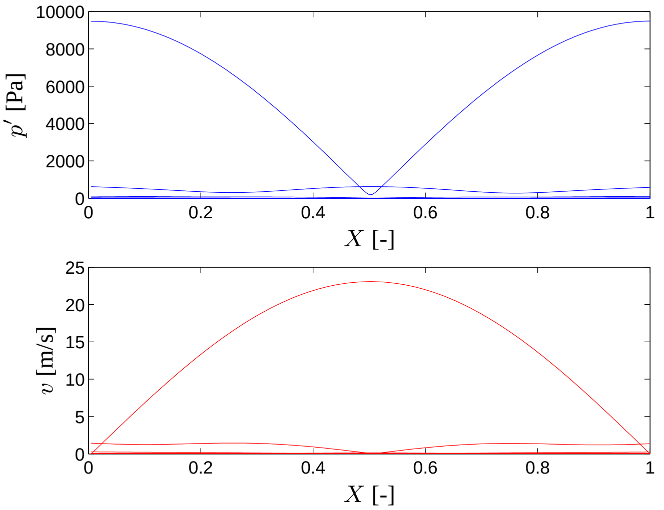

Figure 4: Distribution of acoustic pressure and velocity spectra for $R_0 = R_1 = 1$ for the fundamental harmonics and $R_0 = R_1 = 0$ for the higher ones.

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